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Endogenous rates of time preference in monetary growth models: stability and comparative dynamics

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ENDOGENOUS RATES OF TIME PREFERENCE IN MONETARY GROWTH
MODELS: STABILITY AND COMPARATIVE DYNAMICS

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Endogenous rates of time preference in monetary growth models:

Stability and comparative dynamics

by

Omar M. Abdel-Razeq

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1: INTRODUCTION

Considerable attention has been focused on the issue of money neutrality in models of money and growth. This work provides yet another reason for the standard neutrality result of such models. Endogenizing the rate of time preference in optimal monetary growth models provides a rationalization for the non-neutrality result of the descriptive monetary growth models; the so called "Tobin-effect."

The first models of economic growth did not integrate monetary factors in the explanation of the growth process. Such models are usually termed "real" or "non-monetary" models. They are real in the sense that they concentrate on the real sector and ignore the monetary sector of the economy. Like all growth models, they are of two types; "descriptive" and optimal.

Descriptive growth models make ad-hoc assumptions about agent's savings behavior. Typically they assume that savings are a constant proportion of disposable income and manage to ignore the monetary sector by assuming that accumulating wealth does not affect disposable income. The real growth models of Solow (1956) and Swan (1956) are the first examples of descriptive growth models. In these single asset models, the technology or the production function, the savings proportion, and the growth rate of the population determine the steady state capital-labor ratio.

Optimal real growth models derive savings behavior from explicit optimization techniques by maximizing an intertemporal utility function.

In these models, such as Cass (1965), there is only one asset, capital. Instantaneous utility is derived from consumption only. The quantity of money does not affect wealth and thus is absent from the maximization process. Like the descriptive models, the steady state capital-labor ratio is determined by the production technology and other parameters of the model. Such parameters include the discount rate, the rate of population growth and the rate of capital depreciation.

This separation of the real sector from the monetary sector decomposes the growth process into two subproblems; the determination of the real variables including the growth rate of the economy, and the determination of the monetary variables.

Recently, considerable attention has been focused on the issue of neutrality in models of money and growth. These monetary growth models are also either descriptive or optimal. The early attempts were the descriptive models of Tobin (1965) and Sidrauski (1967a). These models, in common with the descriptive real growth models, make ad-hoc assumptions about the agents' savings behavior. Saving is assumed a constant proportion of disposable income. However, they incorporate the monetary sector into the growth process. Money is introduced as an alternative mean of holding wealth. Thus, agents have two assets in which to hold their wealth; real capital and money. Therefore, the quantity of money in the economy influences wealth and disposable income; hence it affects savings.

The introduction of money in this fashion produced different conclusions from those of the descriptive growth models. The mere existence

of outside money prevents the economy from attaining the Solow steady
state capital-labor ratio. For any savings ratio, the steady state
capital-labor ratio of the descriptive monetary growth models is less
than that of the Solow model.

Another central result of the Tobin (1965) and Sidrauski (1967a)
type models is the long-run non-neutrality of changes in the growth rate
of money. An increase in the growth rate of nominal balances increases
the long-run (steady state) capital-labor ratio. The intuition for this
is simple; an increase in the growth rate of money increases the expected
rate of inflation, driving down the real rate of return on holding money.
In the face of such a change, individuals, with a constant and exogenous
average propensities to save, should be induced to hold greater stocks of
capital. This long-run effect came to be known in the literature as the
"Tobin-effect".

These results indicate that "neglecting the existence of alternative
assets to real capital in the neo-classical model of growth, with saving
being a constant proportion of income, is not a proper way of simplifying
the analysis" (Sidrauski, 1967a, p. 796).

The long-run non-neutrality result was quickly seen to depend
crucially on the descriptive or ad-hoc nature of the savings function,
characteristic of a wide-class of "Keynesian" models. Studies which,
instead, based savings behavior on explicit neoclassical intertemporal
optimization, the optimal monetary growth models, resulted in a long-run
neutrality of changes in the growth rate of money. The seminal work in
this area is that of Sidrauski (1967b). The basic features that

differentiate Sidrauski's optimal monetary growth model from the descrip-
tive models are: the optimization aspect, the infinite planning horizon,
the introduction of real money balances in the instantaneous utility
function, and the discounting of future utilities.

The optimization aspect of the model is in line with the idea of
basing macro models on micro foundations. It provides an "explicit
analysis of individuals' saving behavior, viewed as a process of wealth
accumulation. This is in line with Patinkin's (1965) presentation of the
neoclassical theory of money, and with the Classical Fisherian theory of
saving (1930)" (Sidrauski, 1967b, p. 534).

Real money balances are introduced in the utility function as a
proxy for the services derived from holding money. Agents in the economy
hold money for the services it provides, such as lowering transaction
costs. These services at any point in time are a function of the
quantity of money held. Assuming the functional relation to be a propor-
tional one between the stock of money and its flow benefits, and making
the factor of proportionality equal to unity by a proper choice of units,
real money balances as a stock enter the utility function as flow
services.

The concept of discounting is consistent with the concept of
impatience described by Koopmans (1960), which provides an axiomatic
basis for preferring present to future consumption. The rate of time
preference, or the discount rate, in Sidrauski (1967b) is assumed to be a
positive constant for all points of time.

Beyond these differences, the model deals with the same structural elements as the descriptive money growth model (Sidrauski, 1967a). As far as results are concerned, the basic difference is the super-neutrality character of the optimal model. An increase in the growth rate of nominal money balances does not affect the real sector of the economy in the long-run. The steady-state capital-labor ratio is invariant to changes in the growth rate of money.

In Chapter 2, the equations of Sidrauski (1967b) are derived, the stability of the model is formally analyzed and the long-run neutrality of money is proved. The stability and the long-run results are then illustrated graphically.

Recently, a number of papers with optimal-models have begun to list cases where non-neutrality holds. The finite-horizon, overlapping-generations models of Drazen (1976), Calvo's (1979) analysis of money in the production function, and Brock's (1974) endogenous labor supply are notable examples.

Another trend in the growth literature emphasizes the effect of endogenizing the rate of time preference. Uzawa (1968) analyzed an optimal model of consumption with a variable rate of time preference. Following the Uzawa paper, a number of studies used his approach of endogenizing the rate of time preference; the works of Calvo and Findlay (1978), Findlay (1978), and Obstfeld (1981a) and (1981b) in the field of international trade, and Nairy's (1984) life-cycle consumption are examples.

Uzawa introduces the variability of the rate of time preference by constructing it in such a way that at each point in time, t, it depends

not only on the current utility level but also on utility levels from the start of the planning horizon up to time t. Thus, he assumes that the utility discount factor depends on past utilities as well as present utilities. To illustrate, let $\Delta(t)$ be the utility discount factor and let its dependence on past utilities be given by the functional form

$$\Delta(t) = \int_0^t \delta(U_s) \, ds, \quad \Delta(0) = 0$$

The function $\delta(U)$ is then the agents' rate of time preference, assumed to be an increasing function of utility. That is, $\delta'(U_t)$ is positive, implying that if utility at a future date, say t, increases, then utilities beyond time t, are discounted more. This is so because the utility discount factor is larger now for all times beyond t.

This assumption is not in contradiction with the axiomatic study of Koopmans, Diamond and Williamson (1964). Nairy (1984) shows explicitly the relationship between Uzawa and Koopmans, Diamond, and Williamson (1964). Uzawa's approach in essence imposes "weak separability" on the standard additive utility functional.

Two other approaches have been used to endogenize the rate of time preference. The first uses "recursive" or weakly separable utility functionals as in the paper by Epstein and Hynes (1983). Simply put, weak separability means that the marginal utilities and the marginal rates of substitution depend not only on present values of the arguments but also on their future or past behavior, depending on the formulation used. Epstein and Hynes (1983) discuss the implications of this approach to a

host of different models; the real growth optimal model, a tax incidence model, monetary growth models and others. Their approach is closely related to that of Uzawa (1968). In fact, Epstein and Hynes (1983) point out (page 618) that both approaches are special cases of a general recursive utility functional.

A third approach is that used by Ryder and Heal (1973). In a real growth model, they introduce another argument in the instantaneous utility function. The new variable is a weighted average of past consumption behavior. This can be thought of as the "customary" or "expected" level of consumption. The introduction of this variable in the utility function makes utility dependent on present and past consumption; thus the rate of time preference is made endogenous.

In Chapter 3, Uzawa's approach is used to introduce the variability of the rate of time preference to the optimal monetary growth model of Sidrauski (1967b). It is clear from the analysis of Chapter 2 that, in Sidrauski's model, the constancy of the rate of time preference induces the superneutrality result. The steady-state capital-labor ratio is uniquely determined by the rate of time preference, the rate of capital depreciation, and the growth rate of the population. Since an increase in the growth rate of nominal money balances does not change the rate of time preference, money is superneutral in the long-run. Endogenizing the rate of time preference by making it to be dependent on consumption and real money balances, allows a change in the growth rate of money to affect the rate of time preference, and thus to alter the steady-state capital-labor ratio. Therefore, the variability of the rate of time

preference provides a link between the monetary and the real sectors of the economy.

Chapter 3 presents a derivation of the first-order-conditions for the new model. The stability of the model is then formally analyzed, and the long-run non-neutrality of the model is proved.

Uzawa's approach is chosen over the other two approaches for its relative simplicity and the fact that it maintains the basic structure of the Sidrauski model. This allows one to focus on the effects of endogenizing the rate of time preference.

The relative simplicity of this approach stems from the fact that the standard calculus techniques are still applicable. The other approaches require more complex mathematical techniques and additional assumptions on the structure of the model. For the derivation of the first-order-conditions, variational differentiation (Voltera derivatives) is required; and solving integral equations is needed for stability analysis. When using the recursive functionals approach of Epstein and Hynes (1983), one would come up with two rates of time preference; one depending on future consumption and the other on future money holdings. To avoid the added complexity, Epstein and Hynes have to restrict the analysis along constant paths of consumption and real money balances, in which case the two rates of time preference are equal (Epstein and Hynes, 1983, p. 625).

The complexity of the approach of Ryder and Heal (1973) is augmented by the introduction of one or more state variables. In the real growth

model, one state variable was added to the model. In a monetary growth model, where real money balances are an argument of the utility function, one might need to introduce two state variables; a weighted average of past consumption and a weighted average of past holdings of real money balances. Such additions increase the complexity of the analysis a great deal. Even in the simple case of a one-asset, real growth model, Ryder and Heal restrict their analysis along a constant path of consumption (Ryder and Heal, 1973, p. 5).

Chapter 4 presents a discussion of how comparative dynamics analysis would be applied to the model of Chapter 3. Also, a summary of Fischer's (1979) comparative dynamics analysis of an example of the model of Chapter 2 is presented in the chapter.

Finally, Chapter 5 summarizes the results of Chapters 2 and 3, then provides intuitive explanations for the results concerning the impact of an increase in the growth rate of money on the (long-run) steady state values. Finally, some suggestions for future research are made.

The analytical techniques that will be used include the standard techniques of solving optimal control problems and local stability analysis of differential equations. The first is used in Chapters 2 and 3 to derive the first-order-conditions for the models, while the second is used to analyze the stability of the models.

2: OPTIMAL MONETARY GROWTH

WITH A CONSTANT RATE OF TIME PREFERENCE

Sidrauski (1967b) is the first optimal monetary growth model. All of the work that followed is based on it. However, a detailed derivation of the equations of the model, and a proof of the required stability conditions, and of the short-run and long-run effects of an increase in the growth rate of nominal money balances have not been explicitly done. For our purposes, the most important part of Sidrauski's paper is the macro-model section. The individual's optimization problem provides the basis and rationale for the macro model.

A summary of the model, and a derivation of the stability conditions for the individual's optimization problem are presented in Section 2.1. Section 2.2 presents a derivation of the equations of the macro model and its stability conditions. The macro model is reduced to a three-differential-equations system in per-capita real money balances (m), the capital-labor ratio (k) and real per-capita consumption (c). Then the system is reduced to a two-differential-equations system in c and m. This is in addition to Sidrauski's reduction of the model to two-differential-equations in k and the expected rate of inflation π.

The (k, π) system illustrates the importance of the adaptive expectations hypothesis to the stability of the model. The (c, m) system shows the long-run super neutrality of money graphically. The derivation of the (c, k, m) system follows an approach used by Fischer (1979) and Calvo (1979). This approach provides an alternative way of incorporating

the first-order-conditions into the macro model. Sidrauski derives

demand functions for c and m from the individual's optimality conditions

and incorporates these into the macro model. Alternatively, Fischer

(1979) and Calvo (1979) use the conditions directly to derive a differ-

ential equation in c and thus reduce the model to three differential

equations. This procedure is justified because the first-order-

conditions hold at each instant of time. As will be seen, this approach

is more straight forward and requires less calculations than Sidrauski's

approach. The only difference between Sidrauski's and Fischer's is that

Fischer assumes perfect foresight while Sidrauski assumes adaptive

expectations. Fischer's approach will be used in Chapter 3.

Finally, Section 2.3 presents a proof of the long-run neutrality of

money. This is done using the (c, k, m) and the (c, m) models. The

latter is illustrated graphically.

2.1: Sidrauski's Model and the Individual's Optimization Problem

Sidrauski (1967b) assumes that all individuals are identical, each

has a utility function measuring his welfare at any point in time. This

utility function $U_t = U_t(c_t, m_t)$ has consumption at time t (c_t) and real

money balances (m_t) as its arguments. The values of c_t and m_t are in

per-capita terms. The utility function is assumed to be strictly

concave, in c_t and m_t, with continuous first and second derivatives.

That is, it is assumed that

(1) $U_{cc} < 0, U_{mm} < 0$ and $J = U_{cc}U_{mm} - U_{cm}^2 > 0$

positive marginal utilities are also assumed; i.e., $U_c, U_m > 0$.

Another restriction made on the utility function stems from assuming that both c_t and m_t are not inferior goods;

$$(2) \qquad J_1 = U_{mm} - U_{cm} \frac{U_m}{U_c} < 0 \text{ and } J_2 = U_{cc} \left(\frac{U_m}{U_c}\right) - U_{cm} < 0$$

The individual is assumed to maximize intertemporal wealth (W) where:

$$(3) \qquad W = \int_0^\infty e^{-\delta t} U(c_t, m_t) \, dt$$

where δ, the rate of time preference, is assumed to be a positive constant.

This optimization problem is constrained by a stock constraint and a flow constraint. The stock constraint stems from the fact that the stock of nonhuman wealth at any time is allocated between the only two assets in the model; capital and money. In per capita terms, the stock constraint is:

$$(4) \qquad a_t = k_t + m_t$$

where a_t = per capita stock of nonhuman wealth at time t and

k_t = capital-labor ratio at time t, a choice variable for the individual.

The flow constraint is derived from the assumption that the gross disposable income y_d is allocated between consumption and gross real savings. Per capita gross disposable income is the sum of per capita

output y_t, and the real value of per capita transfers from the government (v_t). v_t is assumed to be financed totally by the creation of money. Gross real savings (S_t) is the sum of gross capital accumulation (i_t) and gross additions to real money balances (x_t).

The above is summarized in the following equations:

(5) $y_d = c_t + S_t$

(6) $y_d = y_t(k_t) + v_t$

(7) $S_t = i_t + x_t$

where $y(k_t)$ is a constant return to scale production function, assumed to be the same for all (the identical) individuals and the economy as a whole and it satisfies the regularity conditions; $y'(k_t) > 0$, $y''(k_t) < 0$ for all k_t and $y(0) = 0$, $y(\infty) = \infty$, $y'(0) = \infty$ and $y'(\infty) = 0$.

Gross capital accumulation (i_t) is the sum of the net change in the capital-labor ratio $(\dot{k}_t = \frac{dk_t}{dt})$ at time t, the replacement of the depreciated capital (μk_t), and the amount of capital accumulation needed to provide the new members of society with the same amount of capital endowment $(n k_t)$. Where μ is the constant instantaneous rate of capital depreciation and n is the constant instantaneous rate of growth in the population. Thus,

(8) $i_t = \dot{k}_t + (\mu + n)k_t$

Similarly,

$$(9) \qquad x_t = \dot{m}_t + (\pi_t + n) m_t$$

The gross additions to real money balances (x_t) are the sum of net additions $(\dot{m})$, the additions needed to keep the expected real values of the money balances intact $(\pi_t \, m_t)$, and the additions needed to endow new members of society by the same real money balances as the old members $(n \, m_t)$; where π_t is the expected rate of inflation at time t.

Thus, using (5-9) the flow constraint can be written as

$$y(k_t) + v_t = c_t + \dot{k}_t + (\mu + n) k_t + \dot{m}_t + (\pi_t + n) m_t$$

or

$$(10) \qquad y(k_t) + v_t - (\pi_t + n) m_t - (\mu + n) k_t - c_t = \dot{k}_t + \dot{m}_t$$

Differentiating (4) with respect to time and substituting in (10) we have the following form of the flow constraint:

$$(11) \qquad \dot{a}_t = y(k_t) + v_t - (\pi_t + n) m_t - (\mu + n) k_t - c_t$$

Now the individual's optimization problem is to maximize (3) subject to (4), (11) and initial conditions. The initial conditions specify the value of wealth the individual holds at time zero, the start of the planning horizon.

2.1.1: Derivation of the first-order-conditions

Instead of using the Calculus of Variations (Euler equations) as Sidrauski does, the standard optimal control method is used to derive the first order conditions for the problem. (Dropping the t subscripts.)

Let $H_v \equiv U(c, m) + \lambda[y(k) + v - (\mu+n)k - (\pi+n)m - c]$

H_v is the so-called present value Hamiltonian. Now, the optimization problem is to maximize H_v subject to the stock constraint (4).

Form the Lagrangian of the problem:

(12) $L = H_v + q[a-k-m]$

and derive the first-order-conditions from (12).

The first order conditions for an interior solution are (besides the constraints (4) and (11)):

$$\frac{\partial L}{\partial c} = \frac{\partial H_v}{\partial c} = U_c - \lambda = 0 \text{ and thus}$$

(13) $U_c = \lambda$

$$\frac{\partial L}{\partial m} = \frac{\partial H_v}{\partial m} - q = U_m - \lambda(\pi+n) - q = 0 \text{ and thus}$$

(14) $U_m = \lambda[\pi + n + \frac{q}{\lambda}] = \lambda[\pi + n + r]$

where $r \equiv q/\lambda$ is the "implicit interest rate".

$$\dot{\lambda} = \delta\lambda - \frac{\partial L}{\partial a} = \delta\lambda - \frac{\partial H_v}{\partial a} - q = \delta\lambda - q = \lambda[\delta - q/\lambda]$$

and thus

(15) $\quad \dfrac{\dot{\lambda}}{\lambda} = \delta - r$

$$\frac{\partial L}{\partial k} = \frac{\partial H_v}{\partial k} - q = \lambda\big[y'(k) - (\mu+n)\big] - q = \lambda\{y' - (\mu+n) - r\} = 0$$

(16) $\quad \therefore \; y'(k) - (\mu+n) = r$

and

(17) $\quad \lim\limits_{t \to \infty} a(t)\lambda(t)e^{-\delta t} = 0$

Condition (17) is the so-called transversality condition of the problem. Thus, for a path to be optimal, it has to satisfy conditions 4, 11, and (13-17).

To analyze the system, Sidrauski uses equations 13, 14, and 16 to solve for demand functions for c, m, and k. These are functions of a, λ and π:

(18) $\quad c = c_1(a, \lambda, \pi)$

(19) $\quad m = m_1(a, \lambda, \pi)$

(20) $\quad k = k_1(a, \lambda, \pi)$

Note that equation (20) is redundant since from the stock constraint k=a-m.

One approach of analyzing the stability of the problem is to follow Sidrauski's way of arriving at equations (18-20). This involves looking at equations (13) and (14) of the first-order-conditions as a separate subproblem giving rise to "demand" functions for c and m as functions of λ, π, and r:

(21) $c = c^0(\lambda, \pi, r)$

(22) $m = m^0(\lambda, \pi, r)$

and thus r is treated as an exogenous variable since it is determined, outside of equations (13) and (14), by equation (16). Then using the method of comparative statics on (13) and (14), the partials of c^0 and m^0 with respect to λ, π, and r are found. Then equation (16) is used together with these partials to find the partials of (18)-(20). This approach is presented in this section below. Another more straight forward approach is presented in Appendix A.

To find the partials of c^0 and m^0, totally differentiate (13) and (14) to get (holding n constant);

(i) $U_{cc} dc^0 + U_{cm} dm^0 = d\lambda$

(ii) $U_{mc} dc^0 + U_{mm} dm^0 = (\pi+n+r)d\lambda + \lambda d\pi + \lambda dr$

Finding $dc^0/d\lambda$ and $dm^0/d\lambda$:

Setting $d\pi = dr = 0$ in (i and ii) and writing the resulting system in matrix form, gives;

$$(iii) \quad \begin{bmatrix} U_{cc} & U_{cm} \\ U_{mc} & U_{mm} \end{bmatrix} \begin{bmatrix} dc^0/d\lambda \\ dm^0/d\lambda \end{bmatrix} = \begin{bmatrix} 1 \\ \pi+n+r \end{bmatrix}$$

Solving for $\left(dc^0/d\lambda, \; dm^0/d\lambda\right)^T$ (T: for transpose) we get,

$$(iv) \quad \begin{bmatrix} dc^0/d\lambda \\ dm^0/d\lambda \end{bmatrix} = \begin{bmatrix} U_{cc} & U_{cm} \\ U_{mc} & U_{mm} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ \pi+n+r \end{bmatrix}$$

or

$$(v) \quad \begin{bmatrix} dc^0/d\lambda \\ dm^0/d\lambda \end{bmatrix} = \frac{1}{U_{cc}U_{mm} - U_{cm}^2} \begin{bmatrix} U_{mm} & -U_{cm} \\ -U_{cm} & U_{cc} \end{bmatrix} \begin{bmatrix} 1 \\ \pi+n+r \end{bmatrix}$$

$$(vi) \quad \therefore dc^0/d\lambda = \frac{U_{mm} - U_{cm}\left(\pi+n+r\right)}{U_{cc} \, U_{mm} - U_{cm}^2} = \frac{U_{mm} - U_{cm}\frac{U_m}{U_c}}{U_{cc}U_{mm} - U_{cm}^2} = \frac{J_1}{J} < 0$$

$\left(\text{since from (13) and (14); } \pi+n+r = \frac{U_m}{U_c}\right)$

and

$$(vii) \quad dm^0/d\lambda = \frac{-U_{cm} + U_{cc}(\pi+n+r)}{U_{cc}U_{mm} - U_{cm}^2} = \frac{U_{cc}\frac{U_m}{U_c} - U_{cm}}{J} = \frac{J_2}{J} < 0$$

Finding $dc^0/d\pi$ and $dm^0/d\pi$:

Setting $d\lambda = dr = 0$ in (i and ii) and solving for $\left(dc^0/d\pi, \; dm^0 d\pi\right)^T$ we get:

$$
\text{(viii)} \quad
\begin{bmatrix} dc^0/d\pi \\ dm^0/d\pi \end{bmatrix}
=
\frac{1}{U_{cc}U_{mm} - U_{cm}^2}
\begin{bmatrix} U_{mm} & -U_{cm} \\ -U_{cm} & U_{cc} \end{bmatrix}
\begin{bmatrix} 0 \\ \lambda \end{bmatrix}
$$

$$
\therefore \; dc^0/d\pi = \frac{-U_{cm}\lambda}{U_{cc}U_{mm}-U_{cm}^2} \; \lessgtr \; 0 \text{ as } U_{cm} \gtrless 0
$$

and

$$
dm^0/d\pi = \frac{U_{cc}\lambda}{U_{cc}U_{mm}-U_{cm}^2} < 0 \quad \left(\text{since } \lambda = U_c > 0\right)
$$

Finding dc^0/dr and dm^0/dr:

Setting $d\lambda = d\pi = 0$ in (i and ii) and solving for $\left(\frac{dc^0}{dr}, \frac{dm^0}{dr}\right)^T$, we get equation (viii) again. Therefore, $dc^0/dr = dc^0/d\pi$ and $dm^0/dr = dm^0/d\pi$.

To arrive at demand functions (18-20), solve (16) for k as a function of r, then substitute that and (22) into the stock constraint and solve for r as a function of a, λ and π:

from (16), $y'(k) - (\mu+n) = r$ implies

(23) $\quad k = k^0(r)$

and from the stock constraint:

(24) $\quad a = k^0(r) + m^0(\lambda, \pi, r)$

since $\frac{dm^0}{dr} < 0$, and since from (23) $\frac{dk^0}{dr} = \frac{1}{y''(k)} < 0$, equation (24) can be solved for r as a function of λ, π and a, that is

(25) $r = r(a, \lambda, \pi)$

From equation (24), find the partials of r with respect to a, λ, and π. First, holding λ and π constant and differentiating (24) gives

$$da = k^{0'}(r)dr + \frac{\partial m^0}{\partial r} dr \text{ and thus}$$

(26) $\frac{\partial r}{\partial a} = \dfrac{1}{k^{0'}(r) + \frac{\partial m^0}{\partial r}} < 0$

similarly,

(27) $\frac{\partial r}{\partial \lambda} = \dfrac{-\partial m^0/\partial \lambda}{k^{0'}(r) + \frac{\partial m^0}{\partial r}} < 0$

and

(28) $\frac{\partial r}{\partial \pi} = \dfrac{-\partial m^0/\partial \pi}{k^{0'}(r) + \frac{\partial m^0}{\partial r}} < 0$

Now substituting (25) into (21), (22) and (23) gives the demand functions (18-20). At this point we are able to find the partials of (18-20) with respect to their arguments.

The partials of c_1 (equation 18):

(18a) $\quad \dfrac{\partial c_1}{\partial \lambda} = \dfrac{\partial c^0}{\partial \lambda} + \dfrac{\partial c^0}{\partial r} \cdot \dfrac{\partial r}{\partial \lambda}$

(18b) $\quad \dfrac{\partial c_1}{\partial a} = \dfrac{\partial c^0}{\partial r} \cdot \dfrac{\partial r}{\partial a}$

(18c) $\quad \dfrac{\partial c_1}{\partial \pi} = \dfrac{\partial c^0}{\partial \pi} + \dfrac{\partial c^0}{\partial r} \cdot \dfrac{\partial r}{\partial \pi}$

The partials of m_1 (equation 19):

(19a) $\quad \dfrac{\partial m_1}{\partial \lambda} = \dfrac{\partial m^0}{\partial \lambda} + \dfrac{\partial m^0}{\partial r} \cdot \dfrac{\partial r}{\partial \lambda}$

(19b) $\quad \dfrac{\partial m_1}{\partial a} = \dfrac{\partial m^0}{\partial r} \cdot \dfrac{\partial r}{\partial a}$

(19c) $\quad \dfrac{\partial m_1}{\partial \pi} = \dfrac{\partial m^0}{\partial \pi} + \dfrac{\partial m^0}{\partial r} \cdot \dfrac{\partial r}{\partial \pi}$

And the partials of k_1 (equation 20)

(20a) $\quad \dfrac{\partial k_1}{\partial \lambda} = k^{0'}(r) \cdot \dfrac{\partial r}{\partial \lambda}$

(20b) $\quad \dfrac{\partial k_1}{\partial a} = k^{0'}(r) \cdot \dfrac{\partial r}{\partial a}$

(20c) $\quad \dfrac{\partial k_1}{\partial \pi} = k^{0'}(r) \cdot \dfrac{\partial r}{\partial \pi}$

Now the stage is set to analyze the stability of the individual's optimization problem.

2.1.2: Stability of the individual's optimization problem

Keeping in mind demand functions 18-20, the model is described by

(15) $\dot{\lambda} = (\delta - r)\lambda$

and

(11) $\dot{a} = y(k) + v - (\mu + n)k - (\pi + n)m - c$

The steady state values of λ and a are the solution of the equations resulting from setting $\dot{\lambda} = 0$ and $\dot{a} = 0$. Denote these steady state values by λ^* and a^*. Therefore, steady state implies

(15a) $\delta = r$

and

(11a) $y(k) + v - (\mu + n)k - (\pi + n)m - c = 0$

where k, m and c are given by (18-20).

To analyze the stability of the system, linearize equations (15) and (11) about (λ^*, a^*). This gives

$$\dot{\lambda} = (\delta - r)(\lambda - \lambda^*) - \lambda^* \frac{\partial r}{\partial \lambda}(\lambda - \lambda^*) - \lambda^* \frac{\partial r}{\partial a}(a - a^*)$$

and since $\delta - r = 0$ at steady state,

(29) $\dot{\lambda} = -\lambda*\dfrac{\partial r}{\partial \lambda}(\lambda-\lambda*) - \lambda*\dfrac{\partial r}{\partial a}(a-a*)$

and from (11)

$$\dot{a} = \left(y'-(\mu+n)\right)\dfrac{\partial k_1}{\partial \lambda}(\lambda-\lambda*) + \left(y'-(\mu+n)\right)\dfrac{\partial k_1}{\partial a}(a-a*)$$

$$- (\pi+n)\dfrac{\partial m_1}{\partial \lambda}(\lambda-\lambda*) - (\pi+n)\dfrac{\partial m_1}{\partial a}(a-a*) - \dfrac{\partial c_1}{\partial \lambda}(\lambda-\lambda*) - \dfrac{\partial c_1}{\partial a}(a-a*)$$

but

(16) $y' - (\mu+n) = r$

Therefore,

(30) $\dot{a} = \left[r\dfrac{\partial k_1}{\partial \lambda} - (\pi+n)\dfrac{\partial m_1}{\partial \lambda} - \dfrac{\partial c_1}{\partial \lambda}\right](\lambda-\lambda*) + \left[r\dfrac{\partial k_1}{\partial a} - (\pi+n)\dfrac{\partial m_1}{\partial a} - \dfrac{\partial c_1}{\partial a}\right](a-a*)$

and in matrix form:

(31) $\begin{bmatrix} \dot{\lambda} \\ \dot{a} \end{bmatrix} = \begin{bmatrix} -\lambda*\dfrac{\partial r}{\partial \lambda} & -\lambda*\dfrac{\partial r}{\partial a} \\ r\dfrac{\partial k_1}{\partial \lambda} - (\pi+n)\dfrac{\partial m_1}{\partial \lambda} - \dfrac{\partial c_1}{\partial \lambda} & r\dfrac{\partial k_1}{\partial a} - (\pi+n)\dfrac{\partial m_1}{\partial a} - \dfrac{\partial c_1}{\partial a} \end{bmatrix} \begin{bmatrix} \lambda-\lambda* \\ a-a* \end{bmatrix}$

Call the two-by-two matrix in (31), D and find its determinant and trace.

$$(32) \quad |D| = -\lambda* \frac{\partial r}{\partial \lambda} \cdot r\frac{\partial k_1}{\partial a} + \lambda*(\pi+n) \frac{\partial r}{\partial \lambda} \frac{\partial m_1}{\partial a} + \lambda*\frac{\partial r}{\partial \lambda} \frac{\partial c_1}{\partial a}$$

$$+ \lambda*r \frac{\partial r}{\partial a} \frac{\partial k_1}{\partial \lambda} - \lambda*(\pi+n) \frac{\partial r}{\partial a} \cdot \frac{\partial m_1}{\partial \lambda} - \lambda* \frac{\partial r}{\partial a} \cdot \frac{\partial c_1}{\partial \lambda}$$

$$= \lambda*[-r \frac{\partial r}{\partial \lambda} \frac{\partial k_1}{\partial a} + (\pi+n) \frac{\partial r}{\partial \lambda} \frac{\partial m_1}{\partial a} + \frac{\partial r}{\partial \lambda} \frac{\partial c_1}{\partial a} + r \frac{\partial r}{\partial a} \frac{\partial k_1}{\partial \lambda}$$

$$- (\pi+n) \frac{\partial r}{\partial a} \frac{\partial m_1}{\partial \lambda} - \frac{\partial r}{\partial a} \frac{\partial c_1}{\partial \lambda}]$$

Using 18a, 18b, 19a, 19b, 20a, and 20b, equation (32) becomes:

$$(33) \quad |D| = \lambda*\{-r \frac{\partial r}{\partial \lambda} k^{0'}(r)\frac{\partial r}{\partial a} + (\pi+n) \frac{\partial r}{\partial \lambda} \cdot \frac{\partial m^0}{\partial r} \cdot \frac{\partial r}{\partial a} + \frac{\partial r}{\partial \lambda} \frac{\partial c^0}{\partial r} \cdot \frac{\partial r}{\partial a}$$

$$+ r \frac{\partial r}{\partial a} k^{0'}(r)\frac{\partial r}{\partial \lambda} - (\pi+n) \frac{\partial r}{\partial a} \cdot (\frac{\partial m^0}{\partial \lambda} + \frac{\partial m^0}{\partial r} \cdot \frac{\partial r}{\partial \lambda})$$

$$- \frac{\partial r}{\partial a} (\frac{\partial c^0}{\partial \lambda} + \frac{\partial c^0}{\partial r} \cdot \frac{\partial r}{\partial \lambda})\}$$

$$= -\lambda*\{(\pi+n) \frac{\partial r}{\partial a} \cdot \frac{\partial m^0}{\partial \lambda} + \frac{\partial r}{\partial a} \frac{\partial c^0}{\partial \lambda}\}$$

$$= -\lambda*\frac{\partial r}{\partial a} \{(\pi+n) \frac{\partial m^0}{\partial \lambda} + \frac{\partial c^0}{\partial \lambda}\}$$

since $\lambda* = U_c > 0$ and $\frac{\partial r}{\partial a} < 0$, the sign of $|D|$ is the same as that of

$\{(\pi+n) \frac{\partial m^0}{\partial \lambda} + \frac{\partial c^0}{\partial \lambda}\}$.

But $\frac{\partial m^0}{\partial \lambda} = \frac{J_2}{J}$ and $\frac{\partial c^0}{\partial \lambda} = \frac{J_1}{J}$,

Therefore, $(\pi+n)\dfrac{\partial m^0}{\partial\lambda} + \dfrac{\partial c^0}{\partial\lambda} = \dfrac{(\pi+n)J_2+J_1}{J}$, and since $J>0$, the sign of $|D|$

is that of $(\pi+n)J_2 + J_1$.

The trace of D, $tr(D)$, is

$$(34) \qquad tr(D) = -\lambda\frac{\partial r}{\partial\lambda} + r\frac{\partial k_1}{\partial a} - (\pi+n)\frac{\partial m_1}{\partial a} - \frac{\partial c_1}{\partial a}$$

Substituting for $\dfrac{\partial r}{\partial\lambda}$, $\dfrac{\partial k_1}{\partial a}$, $\dfrac{\partial m_1}{\partial a}$ and $\dfrac{\partial c_1}{\partial a}$ from equations (27), (20b), (19b),

and (18b); equation (34) becomes,

$$(35) \qquad tr(D) = -\lambda\left(\frac{-\partial m^0/\partial\lambda}{k^{0'}(r)+\frac{\partial m^0}{\partial r}}\right) + rk^{0'}(r)\frac{\partial r}{\partial a} - (\pi+n)\frac{\partial m^0}{\partial r}\frac{\partial r}{\partial a} - \frac{\partial c^0}{\partial r}\cdot\frac{\partial r}{\partial a}$$

but $\dfrac{\partial r}{\partial a} = \dfrac{1}{k^{0'}(r)+\frac{\partial m^0}{\partial r}} < 0$

$\dfrac{\partial m^0}{\partial\lambda} = \dfrac{J_2}{J}$,

$k^{0'}(r) = \dfrac{1}{y''}$,

$\dfrac{\partial m^0}{\partial r} = \dfrac{\lambda U_{cc}}{J}$,

and $\dfrac{\partial c^0}{\partial r} = \dfrac{-\lambda U_{cm}}{J}$

$\therefore$ (35) becomes

$$(36) \quad tr(D) = \left(\frac{\partial r}{\partial a}\right)\{\lambda\frac{J_2}{J} + \frac{1}{r}\frac{1}{y''} - (\pi+n)\left(\frac{\lambda U_{cc}}{J}\right) + \frac{\lambda U_{cm}}{J}\}$$

$$= \left(\frac{\partial r}{\partial a}\right)\{\lambda\frac{J_2}{J} + \frac{r}{y''} - \frac{\lambda\left((\pi+n)U_{cc} - U_{cm}\right)}{J}\}$$

but from the F.O.C., $\pi + n + r = \frac{U_m}{U_c}$ and thus $(\pi+n) = \frac{U_m}{U_c} - r$. Substituting in (36),

$$(37) \quad tr(D) = \frac{\partial r}{\partial a}\{\lambda\frac{J_2}{J} + \frac{r}{y''} - \frac{\lambda\left((\frac{U_m}{U_c} - r)U_{cc} - U_{cm}\right)}{J}\}$$

$$= \frac{\partial r}{\partial a}\{\lambda\frac{J_2}{J} + \frac{r}{y''} - \frac{\lambda\left(\frac{U_m}{U_c}U_{cc} - U_{cm}\right)}{J} + \frac{\lambda r U_{cc}}{J}\}$$

$$= \frac{\partial r}{\partial a}\{\lambda\frac{J_2}{J} + \frac{r}{y''} - \lambda\frac{J_2}{J} + \frac{\lambda r U_{cc}}{J}\}$$

$$= \frac{\partial r}{\partial a} \cdot r \cdot \{\frac{1}{y''} + \frac{\lambda U_{cc}}{J}\}$$

but $\frac{\partial r}{\partial a} < 0$, $r = \delta$ (at steady state) > 0, $\lambda = U_c > 0$, $U_{cc} < 0$, $y'' < 0$ and $J > 0$.

Therefore, the trace is positive. Since the trace is the sum of the two eigen values (roots) of the matrix, the two roots are either both positive or of opposite signs. If they are both positive, then the system is totally unstable (an unstable node). To rule out this possibility, a condition is imposed on the system to make the roots have opposite signs. Since the determinant of D is the product of the two

roots, the condition is such that the determinant is negative. Therefore, a necessary and sufficient condition for saddle-point stability of the system is

$$(38) \quad (\pi + n) \, J_2 + J_1 < 0$$

Condition (38) is satisfied for any expected rate of inflation that is smaller in absolute value than n, given that m and c are not both inferior.

Sidrauski solves equations (15a) and (11a) for λ as a function of a, π and v, and concludes that the demand functions of c, m and k (18-20) become functions of a, π and v. That is the new demand functions are:

$$(39) \quad c = c'(a, \pi, v),$$

$$(40) \quad m = m'(a, \pi, v), \text{ and}$$

$$(41) \quad k = k'(a, \pi, v)$$

However, note that $\lambda(t)$ is a function not only of present π and v but also of π and v of subsequent future dates. One would expect that what people expect the prices to be in subsequent periods (not just next period) to affect decisions made at the present. One way to justify Sidrauski's approach, is to assume that individual's perceive the expected rate of inflation to be the same for all future dates. In which

case $\pi(t)$ for the individual's problem is not only exogenous but constant as well. Similarly for $v(t)$.

2.2: Analysis of the Macro Model

The individual's optimization results form the basis or rationale for the macro model. In the macro model, Sidrauski makes the following assumptions:

I) Expectations are adaptive and thus,

$$(42) \quad \dot{\pi} = b\left(\dot{P}/p - \pi\right); \; b > 0.$$

II) Equilibrium in the money market is such that the demand for money is equal to the total money supply at each moment in time. Thus, given the total money supply M^s, the equilibrium condition can be written as:

$$(43) \quad \frac{M^d}{PN} = \frac{M^s}{PN} = \frac{M}{PN} = m \quad \text{(where N: labor force).}$$

Given the money demand function, the money market equilibrium determines the price level, P, at each point in time.

III) Each unit of the economy receives exactly the same amount of net transfers from the government. These net transfers (v) are assumed to be totally financed by the creation of money. Therefore, the amount of money that is issued per unit of time is equal to v.

That is

$$(44) \qquad v = \frac{\dot{M}}{PN} = \theta m$$

where $\theta \left(= \frac{\dot{M}}{M} \right)$ is the growth rate of nominal money supply, and is assumed to be given and constant.

IV) In addition to assumptions (I-III), previously made assumptions still hold. In particular, $n\left(= \dot{N}/N \right)$, the growth rate of the labor force and μ, the depreciation rate and δ, the rate of time preference are constant.

2.2.1: The macro model in k and π

Since $a = k + m$ and $v = \theta m$, the demand functions (39-41) can be used to derive aggregate demand functions for c and m. These aggregate demands are functions of k, π, and θ. Therefore,

$$(45) \qquad c = c\left(k, \ \theta, \ \pi \right)$$

and

$$(46) \qquad m = m\left(k, \ \theta, \ \pi \right)$$

are the relevant demand functions for the macro model.

Then Sidrauski uses the fact that any portion of net output that is not consumed is necessarily used for capital accumulation to come up with an expression for the rate of change in the capital stock.

(47) $\dot{k} = y(k) - (\mu+n)k - c(k, \theta, \pi)$

To derive a $\dot{\pi}$ equation as a function of k and π, start by rewriting the money market equilibrium condition using the money demand function (46). Then differentiate with respect to time and use the adaptive expectations hypothesis.

Equilibrium in the money market is such that money demand is equal to money supply. Therefore,

(48) $\dfrac{M^d}{PN} = \dfrac{M^s}{PN} \equiv m = m(k, \theta, \pi)$

which by differentiation with respect to time gives (for a given θ)

(49) $m \cdot \left(\theta - \dfrac{\dot{P}}{P} - n\right) = \dfrac{\partial M}{\partial k} \cdot \dot{k} + \dfrac{\partial M}{\partial \pi} \cdot \dot{\pi}$

from the adaptive expectations assumption (42),

(50) $\dfrac{\dot{P}}{P} = \dfrac{\dot{\pi}}{b} + \pi$

substituting in (49)

(51) $m \cdot \left(\theta - \dfrac{\dot{\pi}}{b} - \pi - n\right) = \dfrac{\partial M}{\partial k} \dot{k} + \dfrac{\partial M}{\partial \pi} \cdot \dot{\pi}$

and solving for $\dot{\pi}$,

$$(52) \qquad \dot{\pi} = \frac{1}{\frac{m}{b} + \frac{\partial m}{\partial \pi}} \{ m(\theta - \pi - n) - \frac{\partial m}{\partial k} \dot{k} \}$$

where $\dot{k}$ is given by (47).

Rewriting (52) by factoring out m,

$$(53) \qquad \dot{\pi} = \frac{m}{\frac{m}{b}\left(1 + \frac{b}{m}\frac{\partial m}{\partial \pi}\right)} \{ (\theta - \pi - n) - \frac{1}{m}\frac{\partial m}{\partial k} \dot{k} \}$$

$$= \frac{b}{1 + \frac{b}{m}\frac{\partial m}{\partial \pi}} \{ (\theta - \pi - n) - \frac{1}{m}\frac{\partial m}{\partial k} \dot{k} \}$$

Now equations (47) and (53) comprise the reduced model in k and π. Setting $\dot{\pi} = \dot{k} = 0$, steady-state values of π and k $(\pi^*,\ k^*)$ are the solution to the resulting system. Therefore, $(\pi^*,\ k^*)$ are such that

$$(54) \qquad \pi^* = \theta - n$$

and

$$(55) \qquad c^*(k^*,\ \theta,\ \pi^*) = y(k^*) - (\mu + n)k^*$$

To analyze the stability of the system linearize (47) and (53) about $(\pi^*,\ k^*)$,

$$(56) \qquad \dot{k} = [y'(k^*) - (\mu + n) - \frac{\partial c^*}{\partial k^*}](k - k^*) - \frac{\partial c^*}{\partial \pi^*}(\pi - \pi^*)$$

and

$$(57) \quad \dot{\pi} = \frac{-b(\frac{1}{m^*}\frac{\partial m^*}{\partial k^*})}{1+\frac{b}{m^*}\frac{\partial m^*}{\partial \pi^*}} \left(y'(k^*) - (\mu+n) - \frac{\partial c^*}{\partial k^*}\right)(k-k^*)$$

$$- \frac{b}{1+\frac{b}{m^*}\frac{\partial m^*}{\partial \pi^*}} \left(1 + \frac{1}{m^*}\frac{\partial m^*}{\partial k^*}\left(- \frac{\partial c^*}{\partial \pi^*}\right)\right)(\pi-\pi^*)$$

and in matrix form, (58)

$$
\begin{bmatrix} \dot{k} \\ \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} y'(k^*)-(\mu+n)-\frac{\partial c^*}{\partial k^*} & -\frac{\partial c^*}{\partial \pi^*} \\ \\ \frac{-b(\frac{1}{m^*}\frac{\partial m^*}{\partial k^*})}{1+\frac{b}{m^*}\frac{\partial m^*}{\partial \pi^*}}(y'(k^*)-(\mu+n)-\frac{\partial c^*}{\partial k^*}) & \frac{-b}{1+\frac{b}{m^*}\frac{\partial m^*}{\partial \pi^*}}(1-\frac{1}{m^*}\frac{\partial m^*}{\partial k^*}\frac{\partial c^*}{\partial \pi^*}) \end{bmatrix} \begin{bmatrix} k-k^* \\ \\ \pi-\pi^* \end{bmatrix}
$$

The determinant of the matrix in (58) is

$$(59) \quad \frac{-b}{1+\frac{b}{m^*}\frac{\partial m^*}{\partial \pi^*}} \left(y'(k^*) - (\mu+n) - \frac{\partial c^*}{\partial k^*}\right)\left(1 - \frac{1}{m^*}\frac{\partial m^*}{\partial k^*}\frac{\partial c^*}{\partial \pi^*}\right)$$

$$- \frac{\partial c^*}{\partial \pi^*} \cdot \frac{(b)(\frac{1}{m^*}\frac{\partial m^*}{\partial k^*})}{1+\frac{b}{m^*}\frac{\partial m^*}{\partial \pi^*}} \left(y'(k^*) - (\mu+n) - \frac{\partial c^*}{\partial k^*}\right)$$

$$= \frac{-b}{1+\frac{b}{m^*}\frac{\partial m^*}{\partial \pi^*}} \left(y'(k^*) - (\mu+n) - \frac{\partial c^*}{\partial k^*}\right)$$

and the trace of the matrix is

$$(60) \quad \left(y'(k^*) - (\mu+n) - \frac{\partial c^*}{\partial k^*}\right) - \frac{b}{1+\frac{b}{m^*}\frac{\partial m^*}{\partial \pi^*}} \left(1 - \frac{1}{m^*}\frac{\partial m^*}{\partial k^*}\frac{\partial c^*}{\partial \pi^*}\right)$$

An increase in π reduces the real return on holding money and thus lowering the demand for real money balances. This reduction in demand creates an excess supply in the market for money causing prices to increase. The increase in prices raises the expected rate of inflation (π) which in turn increases prices and so on. Therefore, the steady state quantity of real money balances is smaller as π increases $\left(\frac{\partial m^*}{\partial \pi^*} < 0\right)$. On the other hand, an increase in the capital stock reduces its marginal product, reducing the real rate of return on captial relative to real money balances. This induces an increase in demand for real money balances. Thus, the steady state quantity of real money balances rises with the increase in k $\left(\frac{\partial m^*}{\partial k^*} > 0\right)$. Given the above relations, the trace (60) and the determinant (59) are ambiguous in sign. A necessary and sufficient condition for stability is that the two roots of the matrix have negative real parts. This requires the determinant to be positive, since it is the product of the two roots, and the trace, the sum of the two roots, to be negative. Thus, from (59) and (60), stability of the model requires that the rate of capital accumulation be a decreasing function of capital at steady state;

$$(61) \quad y'(k^*) - (\mu+n) - \frac{\partial c^*}{\partial k^*} < 0$$

and that the expectations adjustment coefficient (b) be restricted such that

$$(62) \quad \frac{1}{1+\frac{b}{m^*}\frac{\partial m^*}{\partial \pi^*}} > 0$$

Since $(y'(k^*) - (\mu+n))$ is positive at steady state (it is equal to

δ), condition (61) requires that c^* be an increasing function of k^*.

This makes sense, since an increase in k raises disposable income which

increases consumption.

Note that condition (62) is less likely to hold, the larger the

expectations adjustment coefficient. However, in case (62) is not

satisfied and (61) still holds, the steady state will be a saddle point.

The determinant becomes negative, and thus only one of the roots will

have a negative real part.

2.2.2: The model in c, k, and m: Fischer's approach

As an alternative approach to going through the first-order-

conditions and deriving demand functions to be used in the macro model,

Fischer (1979) uses the first-order-conditions directly together with the

macro model assumptions to reduce the model to a three differential

equation system. The only difference from Sidrauski's is that Fischer

assumes perfect foresight. Thus, the expected rate of inflation at each

time is equal to the actual rate of inflation. That is

(63) $\pi = \dfrac{\dot{P}}{P}$ for all t.

In this section the model is reduced to a system in c, k, and m.

The $\dot{k}$ equation is similar to that of Sidrauski, equation (47) of the

last section. In Fischer's set up, the $\dot{k}$ equation can actually be

derived from the constraints of the model and the hypothesis of perfect

foresight. This is achieved by solving the stock constraint for k,

differentiating with respect to time and then using the flow constraint to eliminate a.

$$\dot{k} = y(k) + v - (\pi+n)m - (\mu+n)k - c - \dot{m}$$

But v in the macro model is given by (44) and thus

(64) $$\dot{k} = y(k) + (\theta-\pi-n)m - (\mu+n)k - c - \dot{m}$$

and since $m \equiv \frac{M}{PN}$ then

(65) $$\dot{m} = (\theta - \dot{P}/P - n)m$$

and using the perfect foresight hypothesis, (63):

(66) $$\dot{m} = (\theta - \pi - n)m$$

Substituting (66) into (64) gives the final form of the $\dot{k}$ equation:

(67) $$\dot{k} = y(k) - (\mu+n)k - c$$

To make $\dot{m}$ a function of c, k, and m, the first-order-conditions are used to solve for π and then it is substituted out of equation (66).

From (13) and (14),

$$(68) \qquad \frac{U_m}{U_c} = \pi + n + r$$

substituting for r from equation (16) and solving for π,

$$(69) \qquad \pi = \frac{U_m}{U_c} - y'(k) + \mu$$

and now substituting in (66) gives the final form of the $\overset{\bullet}{m}$ equation,

$$(70) \qquad \overset{\bullet}{m} = \left(\theta - \frac{U_m}{U_c} + y'(k) - \mu - n\right)m$$

The $\overset{\bullet}{c}$ equation is derived from the first-order-conditions as follows:

Differentiating equation (13) with respect to time

$$(71) \qquad U_{cc} \overset{\bullet}{c} + U_{cm} \overset{\bullet}{m} = \overset{\bullet}{\lambda}$$

substituting for $\overset{\bullet}{\lambda}$ from equation (15), and using (16) and (13),

$$(72) \qquad U_{cc} \overset{\bullet}{c} + U_{cm} \overset{\bullet}{m} = U_c(\delta - y' + \mu + n)$$

$$(73) \qquad \therefore \overset{\bullet}{c} = \frac{1}{U_{cc}} \{U_c(\delta - y' + \mu + n) - U_{cm} \overset{\bullet}{m}\}$$

Therefore, equations (67), (70) and (73) comprise the reduced model in c, k, and m. Setting $\dot{c}=\dot{m}=\dot{k}=0$ and denoting the solution of the resulting system by $(c*, m*, k*)$; these steady state values of c, m, and k are such that

(73a) $\delta - y'(k*) + \mu + n = 0$

(70a) $\theta - \dfrac{U_m}{U_c} + y'(k*) - \mu - n = 0$

and

(67a) $y(k*) - (\mu + n)k* - c* = 0$

For stability analysis, linearize (73), (70) and (67) about $(c*, m*, k*)$, taking into account (73a), (70a) and (67a). The linearized system is

(74) $\dot{c} = \dfrac{1}{U_{cc}} \{ -U_{cm} \dfrac{J_2}{U_c} m*(c-c*) + U_{cm} \dfrac{J_1}{U_c} m*(m-m*)$

$- y''(k*)(U_c + U_{cm} m*) (k-k*) \}$

(75) $\dot{m} = \dfrac{J_2}{U_c} m*(c-c*) - \dfrac{J_1}{U_c} m*(m-m*) + y''(k*) m*(k-k*) \}$

and

(76) $\dot{k} = (-1)(c-c*) + (0)(m-m*) + [y'(k*) - (\mu + n)](k-k*)$

and in matrix form we have:

$$(77) \quad \begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} \dfrac{-U_{cm}}{U_{cc}} \dfrac{J_2}{U_c} m^* & \dfrac{U_{cm}J_1}{U_{cc}U_c} m^* & \dfrac{-y''(k^*)}{U_{cc}}(U_c+U_{cm}m^*) \\[2ex] \dfrac{J_2}{U_c} m^* & -\dfrac{J_1}{U_c} m^* & y''(k^*)m^* \\[2ex] -1 & 0 & \delta \end{bmatrix} \begin{bmatrix} c-c^* \\ m-m^* \\ k-k^* \end{bmatrix}$$

The determinant of the matrix in (77) is equal to

$$(-1) \quad \begin{vmatrix} \dfrac{U_{cm}}{U_{cc}} \dfrac{J_1}{U_c} m^* & \dfrac{-y''(k^*)}{U_{cc}}(U_c+U_{cm}m^*) \\[2ex] \dfrac{-J_1}{U_c} m^* & y''(k^*)m^* \end{vmatrix} + \delta \begin{vmatrix} \dfrac{-U_{cm}}{U_{cc}} \dfrac{J_2}{U_c} m^* & \dfrac{U_{cm}J_1}{U_{cc}U_c} m^* \\[2ex] \dfrac{J_2}{U_c} m^* & -\dfrac{J_1}{U_c} m^* \end{vmatrix}$$

$$= (-1) \quad [\dfrac{U_{cm}}{U_{cc}} \dfrac{J_1}{U_c} m^{*2} y'' - \dfrac{J_1 m^* y''}{U_{cc}} - \dfrac{J_1 U_{cm} y'' m^{*2}}{U_{cc}U_c}] + \delta \quad [0]$$

$$= \dfrac{J_1 m^* y''(k^*)}{U_{cc}} < 0.$$

This indicates that either one or all three roots have negative real parts. To determine which is the case here, consider the trace of the matrix.

The trace is the sum of the diagonal elements and is equal to

$$(78) \qquad \delta - \dfrac{J_1}{U_c} m^* - \dfrac{U_{cm}}{U_{cc}} \dfrac{J_2}{U_c} m^* = \delta - m^*(\dfrac{J_1}{U_c} + \dfrac{U_{cm}}{U_{cc}} \dfrac{J_2}{U_c})$$

using the definitions of J_1 and J_2 in (78),

$$\text{trace} = \delta - m*\left(\frac{U_{mm} - U_{cm} U_m/U_c}{U_c} + \frac{U_{cm}\left(U_{cc} U_m/U_c - U_{cm}\right)}{U_{cc}U_c}\right)$$

$$= \delta - m*\left(\frac{U_{mm} U_{cc} - U_{cc} U_{cm} U_m/U_c + U_{cm} U_{cc} U_m/U_c - U_{cm}^2}{U_{cc}U_c}\right)$$

$$= \delta - m* \frac{J}{U_{cc}U_c} > 0$$

Since the trace is the sum of the roots of the system, and is positive then at least one of the roots is positive. Therefore, only one of the roots have a negative real part. Thus, the steady-state $\left(c*, m*, k*\right)$ is a saddle point.

2.2.3: The model in c and m

We now reduce the model of section 2.2.2 to a two-differential-equations system in c and m. This is done to illustrate the stability of the system and the long-run effects of an increase in θ graphically. In order to reduce the $\left(c, m, k\right)$ system to a $\left(c, m\right)$ system, set $\dot{k}=0$ and by differentiation find the relation between c and k along $\dot{k}=0$.

Setting $\dot{k}=0$ gives

(79) $y\left(k\right) - \left(\mu+n\right)k - c = 0$

and totally differentiating (79) gives

$$(80) \quad \frac{dk}{dc}\bigg|_{\dot{k}=0} = \frac{1}{y'(k) - (\mu+n)}$$

Equation (80) is now used in conjunction with (70) and (73) to analyze the (c, m) system. Linearizing (70) and (73) about the steady state (c^*, m^*) yields:

$$(81) \quad \dot{c} = \frac{1}{U_{cc}} \{-U_c y''(k)\frac{dk}{dc}\bigg|_{\dot{k}=0} - U_{cm}\frac{J_2}{U_c}m^* - U_{cm}y''(k)\frac{dk}{dc}\bigg|_{\dot{k}=0} m^* \} (c-c^*)$$

$$+ \frac{1}{U_{cc}} \{U_{cm}\frac{J_1}{U_c}m^*\} (m-m^*)$$

and

$$(82) \quad \dot{m} = \{\frac{J_2}{U_c}m^* + y''(k)\frac{dk}{dc}\bigg|_{\dot{k}=0}m^* \} (c-c^*) - \frac{J_1}{U_c}m^*(m-m^*)$$

where all the functions in (81) and (82) are evaluated at the steady state. From (80), at steady state

$$(83) \quad \frac{dk}{dc}\bigg|_{\dot{k}=0} = \frac{1}{\delta}$$

Using (83), the linearized system becomes;

$$(84) \quad \dot{c} = \frac{-1}{U_{cc}} \left\{ \frac{U_c y''(k*)}{\delta} + \frac{U_{cm} y''(k*) m*}{\delta} + \frac{U_{cm} J_2}{U_c} m* \right\} (c-c*)$$

$$+ \frac{1}{U_{cc}} \left\{ U_{cm} \frac{J_1}{U_c} m* \right\} (m-m*)$$

and

$$(85) \quad \dot{m} = \left\{ \frac{J_2}{U_c} m* + \frac{y''(k*) m*}{\delta} \right\} (c-c*) - \frac{J_1}{U_c} m* (m-m*)$$

In matrix form,

$$(86) \quad \begin{bmatrix} \dot{c} \\ \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \frac{-1}{U_{cc}} \left\{ \frac{U_c y''(k*)}{\delta} + \frac{U_{cm} y''(k*) m*}{\delta} + \frac{U_{cm} J_2 m*}{U_c} \right\} & \frac{U_{cm} J_1 m*}{U_{cc} U_c} \\ \\ \frac{J_2}{U_c} m* + \frac{y''(k*) m*}{\delta} & -\frac{J_1}{U_c} m* \end{bmatrix} \begin{bmatrix} c-c* \\ \\ m-m* \end{bmatrix}$$

The determinant of the matrix in (86) is equal to

$$\frac{J_1}{U_c} \frac{m*}{U_{cc}} \left\{ \frac{U_c y''}{\delta} + \frac{U_{cm} y'' m*}{\delta} + \frac{U_{cm} J_2 m*}{U_c} \right\} - \frac{U_{cm}}{U_{cc}} \frac{J_1 m*}{U_c} \left\{ \frac{J_2}{U_c} m* + \frac{y'' m*}{\delta} \right\}$$

$$= \frac{J_1 m* y''}{U_{cc} \delta} + \frac{J_1 U_{cm} y'' m*^2}{U_c U_{cc} \delta} + \frac{U_{cm} m*^2 J_1 J_2}{U_c^2 U_{cc}} - \frac{U_{cm} m*^2 J_1 J_2}{U_c^2 U_{cc}}$$

$$- \frac{U_{cm} y'' J_1 m*^2}{U_{cc} U_c \delta} = \frac{J_1 m* y''(k*)}{U_{cc} \delta} < 0$$

and thus the steady state of the system is a saddle point.

To illustrate the saddle point characteristic of the system graphically, the phase diagram in a (c, m) space is constructed. This involves drawing the demarcation curves of the system. First setting $\dot{c}=\dot{m}=0$ (keeping in mind that $\dot{k}$ is also set equal to zero) results in the system whose solution is the steady state values of c and m:

(87) $\delta - y'(k*) + \mu + n = 0$

and

(88) $\theta - \dfrac{U_m}{U_c} + y'(k*) - \mu - n = 0$

Equations (87) and (88) are the demarcation curves of the phase diagram. The curve of equation (87) will be referred to as the $\dot{c}=0$ line. However, it is actually the $\dot{c}=\dot{m}=\dot{k}=0$ line. Similarly, the curve of (88) will be referred to as the $\dot{m}=0$ line.

Equation (87) is independent of m, and thus the $\dot{c}=0$ line is vertical in the (c, m) space as shown in Figure 2.1.

On the other hand, differentiating (88) to find the slope of the $\dot{m}=0$ line in the (c, m) space gives;

$$-\frac{J_1}{U_c} dm + [\frac{J_2}{U_c} + \frac{y''(k)}{y'-(\mu+n)}]\, dc = 0$$

and thus,

$$\left.\frac{dm}{dc}\right|_{\dot{m}=0} = \frac{(J_2/U_c) + y''(k)/(y'-(\mu+n))}{J_1/U_c}$$

which is, generally, ambiguous in sign. However, since at the steady

state $y'-(\mu+n)=\delta > 0$, the $\dot{m}=0$ line is upward sloping as it passes the steady state, S. For simplicity, it is drawn upward sloping throughout in Figure 2.1.
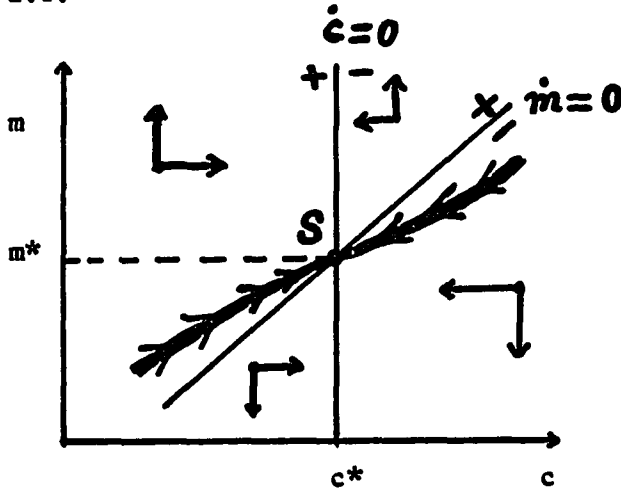
Figure 2.1. Stability of the system

The (+) and (-) signs by the lines in the figure indicate the directions of motion for c and m. These are derived from equations (70) and (73). Starting at a point along the $\dot{m}=0$ line and increasing m causes $\dot{m}$ to become positive,

$$\frac{\partial \dot{m}}{\partial m} = \frac{-J_1}{U_c} m > 0$$

and thus the (+) sign above the $\dot{m}=0$ line and the (-) below it.

Similarly, starting at a point along the $\dot{c}=0$ line and increasing c causes $\dot{c}$ to become negative, and thus the (-) sign to the right of the $\dot{c}=0$ line and the (+) to the left of it.

The directions of motion are shown on the graph by the arrowed right angles. These clearly indicate saddle point behavior. The optimal path is shown by the arrowed heavy line.

2.3: The Long-Run Neutrality of Money

The reduced model of Section 2.2.2 is used to prove the long run neutrality of money. This is done by applying the method of comparative statics on the steady-state of the system, since long-run effects have to do with effects on steady-state values. The steady-state values of c, m and k, (c^*, m^*, k^*), are the solution to the system of equations that results when setting $\dot{c}=0$, $\dot{m}=0$, and $\dot{k}=0$.

From equation (73), $\dot{c}=0$ implies:

$$(89) \quad U_c \left(\delta - y' + \mu + n\right) - U_{cm} \left(\theta - \frac{U_m}{U_c} + y' - \left(\mu + n\right)\right) m = 0$$

from equation (70), $\dot{m}=0$ implies:

$$(90) \quad \theta - \frac{U_m}{U_c} + y' - \left(\mu + n\right) = 0$$

and from equation (67), $\dot{k}=0$ implies:

$$(91) \quad y(k) - \left(\mu + n\right)k - c = 0$$

differentiating (89)-(91) with respect to θ and arranging in matrix form we get:

$$(92) \quad \begin{bmatrix} \dfrac{-U_{cm}J_2}{U_c}m^* & \dfrac{U_{cm}}{U_c}J_1m^* & -y''(U_c+U_{cm}m^*) \\[2ex] J_2/U_c & -J_1/U_c & y'' \\[2ex] -1 & 0 & \delta \end{bmatrix} \begin{bmatrix} dc^*/d\theta \\[2ex] dm^*/d\theta \\[2ex] dk^*/d\theta \end{bmatrix} = \begin{bmatrix} U_{cm}m^* \\[2ex] -1 \\[2ex] 0 \end{bmatrix}$$

or letting x be the matrix on the left-hand side of (92),

$$(93) \quad x \cdot \begin{bmatrix} dc^*/d\theta \\[2ex] dm^*/d\theta \\[2ex] dk^*/d\theta \end{bmatrix} = \begin{bmatrix} U_{cm}m^* \\[2ex] -1 \\[2ex] 0 \end{bmatrix}$$

The determinant of x, $|x|$, is equal to $y''J_1$ and thus is positive. Using Cramer's rule, one can solve for $\dfrac{dc^*}{d\theta}$, $\dfrac{dm^*}{d\theta}$ and $\dfrac{dk^*}{d\theta}$ as follows:

$$(94) \quad \frac{dc^*}{d\theta} = \frac{\begin{vmatrix} U_{cm}m^* & \dfrac{U_{cm}J_1}{U_c}m^* & -y''(U_c+U_{cm}m^*) \\[2ex] -1 & -J_1/U_c & y'' \\[2ex] 0 & 0 & \delta \end{vmatrix}}{|x|}$$

$$= \frac{\delta \begin{vmatrix} U_{cm}m^* & \dfrac{U_{cm}J_1}{U_c}m^* \\[2ex] -1 & -J_1/U_c \end{vmatrix}}{|x|}$$

$$= \frac{\delta \left[\dfrac{-J_1U_{cm}m^*}{U_c} + \dfrac{U_{cm}J_1m^*}{U_c} \right]}{|x|} = 0$$

$$(95) \quad \frac{dm^*}{d\theta} = \frac{\begin{vmatrix} \dfrac{-U_{cm}J_2}{U_c}m^* & U_{cm}m^* & -y''(U_c+U_{cm}m^*) \\[2mm] J_2/U_c & -1 & y'' \\[1mm] -1 & 0 & \delta \end{vmatrix}}{|x|}$$

$$= \frac{(-1)\begin{vmatrix} U_{cm}m^* & -y''(U_c+U_{cm}m^*) \\[1mm] -1 & y'' \end{vmatrix} + \delta \begin{vmatrix} \dfrac{-U_{cm}J_2m^*}{U_c} & U_{cm}m^* \\[2mm] J_2/U_c & -1 \end{vmatrix}}{|x|}$$

$$= \frac{(-1)\left(U_{cm}m^*y'' - y''U_c - y''U_{cm}m^*\right) + \delta\left(\dfrac{U_{cm}J_2m^*}{U_c} - \dfrac{U_{cm}m^* J_2}{U_c}\right)}{|x|}$$

$$= \frac{y''U_c}{|x|} = \frac{U_c}{J_1} < 0$$

and finally,

$$(96) \quad \frac{dk^*}{d\theta} = \frac{\begin{vmatrix} \dfrac{-U_{cm}J_2}{U_c}m^* & \dfrac{U_{cm}J_1m^*}{U_c} & U_{cm}m^* \\[2mm] J_2/U_c & -J_1/U_c & -1 \\[1mm] -1 & 0 & 0 \end{vmatrix}}{|x|}$$

$$= \frac{(-1)\begin{vmatrix} \dfrac{U_{cm}J_1m^*}{U_c} & U_{cm}m^* \\[2mm] -J_1/U_c & -1 \end{vmatrix}}{|x|} = \frac{(-1)\left[\dfrac{-U_{cm}J_1m^*}{U_c} + \dfrac{U_{cm}m^*J_1}{U_c}\right]}{|x|} = 0$$

Thus, equations (94), (95) and (96) show that an increase in θ will only alter the steady state value of real money balances; they are smaller with the higher θ. The real sector, on the other hand, is not affected in the long-run by the increase in θ; c^* and k^* stay the same. Therefore, money is neutral in the long-run.

From equation (66), the original form of the $\dot{m}$ equation, $\dot{m}=0$ implies that $(\theta-\pi-n=0)$ and thus the steady-state value of the expected rate of inflation, π^*, is equal to the rate of money growth minus the rate of population growth;

$$(97) \qquad \pi^* = \theta-n$$

Therefore, an increase in θ will, in the long run, increase the expected rate of inflation by the same amount $\left(\frac{d\pi^*}{d\theta} = 1\right)$.

To illustrate the long run neutrality of money graphically, the phase diagram of Figure 2.1 is used. The effect of an increase in θ on the $\dot{m}=0$ and the $\dot{c}=0$ lines is found from equations (87) and (88). Holding c constant, and differentiating (88) gives:

$$\frac{dm}{d\theta}\bigg|_{\dot{m}=0} = \frac{1}{J_1/U_c} = \frac{U_c}{J_1} < 0$$

This implies that at each level of c, real money balances are smaller, with the higher θ, along the $\dot{m}=0$ line. Therefore, the $\dot{m}=0$ line shifts down to the right, as shown in Figure 2.2. From equation (87), it is clear that the $\dot{c}=0$ line is independent of θ. Thus, the $\dot{c}=0$ line stays the same as θ rises.
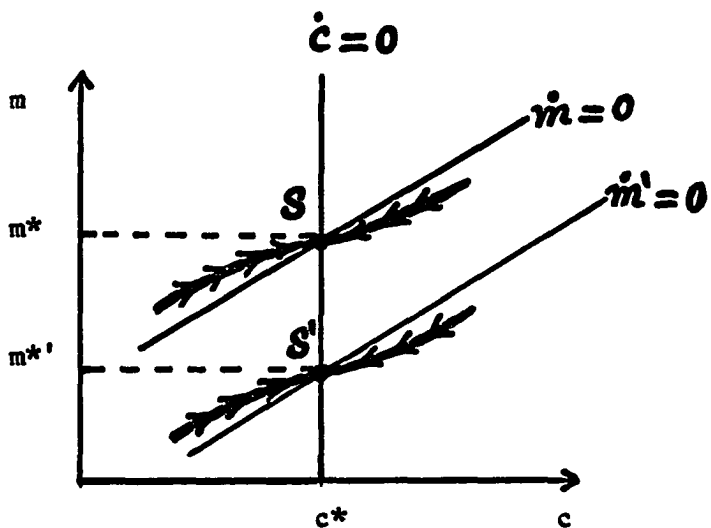
Figure 2.2. The long-run effects of an increase in θ

Therefore, the effect of an increase in θ is to shift the steady-state from point S to point S'. The new steady-state value of real money balances (m*') is smaller, but the steady-state value of consumption (and thus the capital labor ratio) stays the same.

3: OPTIMAL MONETARY GROWTH

WITH A VARIABLE RATE OF TIME PREFERENCE

In this chapter, the rate of time preference is endogenized using Uzawa's approach. Section 3.1 introduces the variability of the rate of time preference to the Sidrauski model and presents a derivation of the first-order-conditions for the new model. The endogenization of the rate of time preference makes the problem a two-state-variable one. Thus, the macro model in this case will be represented by four differential equations as opposed to the three differential equations model of Chapter 2. Section 3.2 derives the macro model and analyzes its stability. In Section 3.3, the long-run effects of an increase in θ are analyzed. Finally, Section 3.4 comments on a transformation that has been applied in the literature to similar models. Such a transformation is used to reduce the dimensionality of the problem to a one-state-variable one. However, it will be shown that the transformation cannot be applied in this case, and that it has been wrongly applied to similar control problems. Also, in this section the conditions under which the transformation will be correct are derived. Thus, applying it to models that meet these conditions is correct and simplifies the analysis a great deal.

3.1: Endogenization of the Rate of Time Preference

and the First-Order-Conditions

Following Uzawa (1968), let $\Delta(t)$ be the utility discount factor at time t, and assume that its functional relation with past utilities is

given by

(1) $\Delta(t) = \int_0^t \delta(U) \, d\tau$ and $\Delta(0) = 0$

and therefore

(2) $\dot{\Delta}(t) = \delta(U)$

The function $\delta(U)$ is assumed to be positive and convex in U. That is

(3) $\delta(U) > 0$, $\delta'(U) > 0$ and $\delta''(U) \geq 0$.

Now the problem of the agent becomes

(4) Max $\int_0^\infty U(c_t, m_t) \, e^{-\Delta(t)} dt$

Subject to

(5) the stock constraint: $a_t = k_t + m_t$

(6) the flow constraint: $\dot{a}_t = y(k_t) + v_t - (\mu+n)k_t - (\pi_t+n) \, m_t - c_t$

(2) the discount factor constraint: $\dot{\Delta}(t) = \delta(U(c_t, m_t))$

and the given initial conditions, $a(0) = a_0$ and $\Delta(0) = 0$.

To find the first-order-conditions, solve (5) for k_t, substitute into (6), and form the Hamiltonian of the problem; (dropping subscripts)

(7) $H = e^{-\Delta}U(c, m) + \lambda[y(a-m) + v - (\mu+n)(a-m) - (\pi+n)m-c] + \gamma\delta(U(c, m))$

The first-order-conditions are:

$$\frac{\partial H}{\partial c} = e^{-\Delta}U_c - \lambda + \gamma\delta'U_c = 0$$

thus,

(8) $\qquad U_c = \dfrac{\lambda}{e^{-\Delta}+\gamma\delta'}$

$$\frac{\partial H}{\partial m} = e^{-\Delta}U_m - \lambda(y'+\pi-\mu) + \gamma\delta'U_m = 0$$

thus,

(9) $\qquad U_m = \dfrac{\lambda}{e^{-\Delta}+\gamma\delta'} (y'+\pi-\mu)$

and the co-state equations;

(10) $\qquad \dot{\lambda} = \dfrac{-\partial H}{\partial a} = -\lambda(y'-(\mu+n))$

and

(11) $\qquad \dot{\gamma} = \dfrac{-\partial H}{\partial \Delta} = e^{-\Delta}U$

and the constraints;

(12) $\qquad \dot{a} = \dfrac{\partial H}{\partial \lambda} = y(a-m) + v - (\mu+n)(a-m) - (\pi+n)m - c$

and

(2) $\qquad \dot{\Delta} = \dfrac{\partial H}{\partial \gamma} = \delta(U(c, m))$.

The conditions can be rewritten in a more convenient way by defining the present value co-state variables $\sigma(t)$ and $\phi(t)$ as follows:

(13) $\quad \sigma(t) \equiv e^{\Delta}\lambda(t)$

and

(14) $\quad \phi(t) \equiv e^{\Delta}\gamma(t)$

Using equation (2), equation (13) implies,

(15) $\quad \dot{\sigma}(t) = e^{\Delta}\dot{\lambda}(t) + e^{\Delta}\lambda(t)\dot{\Delta} = e^{\Delta}\dot{\lambda} + \delta\sigma$

and similarly, equation (14) implies,

(16) $\quad \dot{\phi} = e^{\Delta}\dot{\gamma} + \delta\phi$

Equations (13)-(16), imply that the optimal solution is a path of c, k, m, σ and ϕ that satisfies the following conditions;

(17) $\quad U_c = \dfrac{\sigma}{1+\phi\delta}$

(18) $\quad U_m = \dfrac{\sigma}{1+\phi\delta} (y'+\pi-\mu)$

(19) $\quad \dot{\sigma} = \sigma(\delta-y'+\mu+n)$

(20) $\quad \dot{\phi} = \delta\phi + U$

(12) $\dot{a} = y(a-m) + v - (\mu+n)(a-m) - (\pi+n)m - c$

in addition to the initial conditions and the so-called transversality

conditions;

(21) $\lim_{t->\infty} e^{-\Delta(t)}\sigma(t) \, a(t) = 0$

and

(22) $\lim_{t->\infty} e^{-\Delta(t)}\phi(t) \, \Delta(t) = 0$

Note that if δ were constant and thus $\delta'=0$, then the above

conditions are equivalent to those of the Sidrauski model. Also note

that (17) and (18) yield the same static maximization condition as in the

Sidrauski model;

(23) $\dfrac{U_m}{U_c} = y' + \pi - \mu$

3.2: The Macro Model and Stability

In this section, we maintain the assumptions made in Fischer's model

in Section 2.2.2. The money market is in equilibrium at all t; the

supply of real money balances is equal to the demand for real money

balances. Net government transfer payments (v) are proportional to real

money balances and totally financed by the creation of money. And the

assumption of perfect foresight implying that the expected inflation rate

is equal to the actual inflation rate at each point in time.

Therefore, the $\dot{k}$ and $\dot{m}$ equations are exactly the same as those of Chapter 2, Section 2.2.2. Namely,

(24) $\dot{k} = y(k) - (\mu+n)k - c$

and

(25) $\dot{m} = m\left(\theta - \dfrac{U_m}{U_c} + y' - \mu - n\right)$

The other two equations of the system are those of $\dot{\sigma}$ and $\dot{c}$. The $\dot{\sigma}$ equation is given by (19), and the $\dot{c}$ equation is derived from the first-order-conditions as follows:

Differentiating (17) with respect to time,

(26) $U_{cc}\,\dot{c} + U_{cm}\,\dot{m} = \dfrac{\dot{\sigma}}{1+\phi\delta'} - \dfrac{\sigma(\dot{\phi}\delta'+\dot{\delta}'\phi)}{(1+\phi\delta')^2}$

but

(19) $\dot{\sigma} = \sigma(\delta-y'+\mu+n)$

(20) $\dot{\phi} = \delta\phi + U$

(27) $\dot{\delta}' = \delta''U_c\,\dot{c} + \delta''U_m\,\dot{m}$

and solving (17) for ϕ gives,

$$(28) \quad \phi = \frac{\sigma - U_c}{\delta' U_c}$$

Substituting in (26),

$$(29) \quad U_{cc} \dot{c} + U_{cm} \dot{m} = U_c (\delta - y' + \mu + n)$$

$$- \frac{U_c^2}{\sigma} \{\frac{\delta(\sigma - U_c)}{U_c} + \delta' U + \delta''(\frac{\sigma - U_c}{\delta' U_c}) U_c \dot{c} + \delta''(\frac{\sigma - U_c}{\delta' U_c}) U_m \dot{m}\}$$

Dividing through by U_c and multiplying by σ, and grouping terms;

$$(30) \quad [\frac{\sigma U_{cc}}{U_c} + (\frac{\sigma - U_c}{\delta'})\delta'' U_c]\dot{c} = \sigma(\delta - y' + \mu + n)$$

$$- [\frac{\sigma U_{cm}}{U_c} + (\frac{\sigma - U_c}{\delta'}) \delta'' U_m]\dot{m} - \delta(\sigma - U_c) - \delta' U U_c$$

Let

$$(31) \quad G \equiv \frac{\sigma U_{cc}}{U_c} + (\frac{\sigma - U_c}{\delta'}) \delta'' U_c$$

and

$$(32) \quad x = - [\frac{\sigma U_{cm}}{U_c} + (\frac{\sigma - U_c}{\delta'}) \delta'' U_m]$$

Thus, the $\dot{c}$ equation is given by;

$$(33) \quad \dot{c} = \frac{1}{G}\{x\dot{m} + \sigma(\delta - y' + \mu + n) - \delta(\sigma - U_c) - \delta' U U_c\}$$

which by rewriting gives;

(34) $\dot{c} = \frac{1}{G}\{x\dot{m} - \sigma(y'-\mu-n) + \delta U_c - \delta'UU_c\}$

Therefore, the macro model is characterized by the four differential equations:

(34) $\dot{c} = \frac{1}{G}\{x\dot{m} - \sigma(y'-\mu-n) + \delta U_c - \delta'UU_c\}$

(25) $\dot{m} = m(\theta - \frac{U_m}{U_c} + y' - \mu - n)$

(24) $\dot{k} = y(k) - (\mu+n)k - c$

and

(19) $\dot{\sigma} = \sigma(\delta-y'+\mu+n)$

3.2.1: The stability of the model

The steady state of the system, $(c^*, m^*, k^*, \sigma^*)$, is such that $\dot{c}=\dot{m}=\dot{k}=\dot{\sigma}=0$, which implies that at steady state,

(35) $-\sigma(y'-\mu-n) + \delta U_c - \delta'UU_c = 0$

(36) $\theta - \frac{U_m}{U_c} + y' - \mu - n = 0$

(37) $y(k) - (\mu+n)k - c = 0$

and

(38) $\delta - y' + \mu + n = 0$

where all the functions are evaluated at the steady state (c*, m*, k*, σ*). Note that system (35)-(38) has a unique solution given the properties of the functions.

To analyze the stability of the system, linearize (34), (25), (24), and (19) around (c*, m*, k*, σ*), taking (35)-(38) into account.

Linearizing (34) gives,

$$\dot{c} = \frac{1}{G}\{\frac{xmJ_2}{U_c} + U_{cc}(\delta-\delta'U) - \delta''UU_c^2\} \ (c-c*)$$

$$+ \frac{1}{G}\{\frac{-xmJ_1}{U_c} + U_{cm}(\delta-\delta'U) - \delta''UU_cU_m\} \ (m-m*)$$

$$+ \frac{1}{G}\{xmy'' - \sigma y''\} \ (k-k*)$$

$$- \frac{\delta}{G} \ (\sigma-\sigma*)$$

where $J_1 = U_{mm} - U_{cm}\frac{U_m}{U_c} < 0$ and $J_2 = U_{cc}\frac{U_m}{U_c} - U_{cm} < 0$

For convenience when writing the linearized system in matrix form, let

$$(39) \qquad Z_1 \equiv \frac{1}{G}\{\frac{xmJ_2}{U_c} + U_{cc}(\delta-\delta'U) - \delta''UU_c^2\}$$

$$(40) \qquad Z_2 \equiv \frac{1}{G}\{\frac{-xmJ_1}{U_c} + U_{cm}(\delta-\delta'U) - \delta''UU_cU_m\}$$

and

$$(41) \qquad Z_3 \equiv \frac{1}{G}\{xmy'' - \sigma y''\}$$

linearizing (25) gives,

$$(42) \qquad \dot{m} = \frac{mJ_2}{U_2} (c-c^*) - \frac{mJ_1}{U_c} (m-m^*) + my'' (k-k^*)$$

linearizing (24) gives,

$$(43) \qquad \dot{k} = - (c-c^*) + \delta(k-k^*)$$

and linearizing (19) gives,

$$(44) \qquad \dot{\sigma} = \sigma\delta'U_c(c-c^*) + \sigma\delta'U_m(m-m^*) - \sigma y''(k-k^*)$$

Thus, the linearized system is,

$$(45) \qquad
\begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \\ \dot{\sigma} \end{bmatrix}
=
\begin{bmatrix}
Z_1 & Z_2 & Z_3 & -\dfrac{\delta}{G} \\
\dfrac{mJ_2}{U_c} & \dfrac{-mJ_1}{U_c} & my'' & 0 \\
-1 & 0 & \delta & 0 \\
\sigma\delta'U_c & \sigma\delta'U_m & -\sigma y'' & 0
\end{bmatrix}
\begin{bmatrix} c-c^* \\ m-m^* \\ k-k^* \\ \sigma-\sigma^* \end{bmatrix}$$

where all the elements in the matrix are evaluated at the steady state.

The determinant of the matrix in (45) is,

$$
\frac{\delta}{G}
\begin{vmatrix}
\dfrac{mJ_2}{U_c} & \dfrac{-mJ_1}{U_c} & my'' \\
-1 & 0 & \delta \\
\sigma\delta'U_c & \sigma\delta'U_m & -\sigma y''
\end{vmatrix}
= \frac{\delta}{G} \left\{ 1
\begin{vmatrix}
\dfrac{-mJ_1}{U_c} & my'' \\
\sigma\delta'U_m & -\sigma y''
\end{vmatrix}
- \delta
\begin{vmatrix}
\dfrac{mJ_2}{U_c} & \dfrac{-mJ_1}{U_c} \\
\sigma\delta'U_c & \sigma\delta'U_m
\end{vmatrix}
\right\}
$$

$$
= \frac{\delta}{G}\left\{ \frac{\sigma m J_1 y''}{U_c} - \sigma\delta'mU_m y'' - \delta\left(\frac{\sigma m\delta'U_m J_2}{U_c} + \sigma\delta'mJ_1\right)\right\}
$$

Since J_1, J_2 and $y'' < 0$ and σ, δ, δ', m, U_m, and $U_c > 0$, the determinant is of the same sign as the steady state value of G.

Recall that G is defined by (31) as

$$
(31) \qquad G = \frac{\sigma U_{cc}}{U_c} + \left(\frac{\sigma - U_c}{\delta'}\right)\delta''U_c
$$

But at steady state, $\phi = -U/\delta$ and thus,

$$
U_c = \frac{\sigma}{1 - (\delta'/\delta)U}
$$

which implies that $\sigma - U_c < 0$, and thus, from (31), G is negative at the steady state.

Therefore, the determinant is negative, which (in a four by four system) is necessary and sufficient for the steady state to be a saddle point. This is so because the determinant is the product of the four roots of the system, and thus being negative implies that there is either

one negative root or three negative roots of the system, implying saddle point behavior. In Appendix B, it is shown that the system has a unique negative root.

3.3: The Long-Run Effects of an Increase in θ

The effect of an increase in θ on the steady state values ($c*$, $m*$, $k*$, $\sigma*$) is determined by applying comparative statics to equations (35)-(38); from (35),

$$(35') \quad [U_{cc}(\delta-\delta'U) - \delta''UU_c^2]\frac{dc*}{d\theta} + [U_{cm}(\delta-\delta'U) - \delta''UU_cU_m]\frac{dm*}{d\theta} - \sigma y''\frac{dk*}{d\theta}$$

$$- \delta\frac{d\sigma*}{d\theta} = 0$$

from (36)

$$(36') \quad \frac{J_2}{U_c}\frac{dc*}{d\theta} - \frac{J_1}{U_c}\frac{dm*}{d\theta} + y''\frac{dk*}{d\theta} = -1$$

from (37)

$$(37') \quad -\frac{dc*}{d\theta} + \delta\frac{dk*}{d\theta} = 0$$

and from (38)

$$(38') \quad \delta'U_c\frac{dc*}{d\theta} + \delta'U_m\frac{dm*}{d\theta} - y''\frac{dk*}{d\theta} = 0$$

In matrix form,

$$(46) \quad \begin{bmatrix} U_{cc}(\delta-\delta'U)-\delta''UU_c^2 & U_{cm}(\delta-\delta'U)-\delta''UU_cU_m & -\sigma y'' & -\delta \\ J_2/U_c & -J_1/U_c & y'' & 0 \\ -1 & 0 & \delta & 0 \\ \delta'U_c & \delta'U_m & -y'' & 0 \end{bmatrix} \begin{bmatrix} \dfrac{dc^*}{d\theta} \\ \dfrac{dm^*}{d\theta} \\ \dfrac{dk^*}{d\theta} \\ \dfrac{d\sigma^*}{d\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

where all the functions are evaluated at steady state.

Let the matrix of (46) be F, then the determinant of F is,

$$|F| = \delta \begin{vmatrix} J_2/U_c & -J_1/U_c & y'' \\ -1 & 0 & \delta \\ \delta'U_c & \delta'U_m & -y'' \end{vmatrix}$$

$$= \delta \left\{ \begin{vmatrix} -J_1/U_c & y'' \\ \delta'U_m & -y'' \end{vmatrix} - \delta \begin{vmatrix} J_2/U_c & -J_1/U_c \\ \delta'U_c & \delta'U_m \end{vmatrix} \right\}$$

$$= \delta\left\{ \frac{J_1 y''}{U_c} - \delta'U_m y'' - \delta\left(\frac{\delta'J_2U_m}{U_c} + \delta'J_1\right) \right\} > 0$$

Using Cramer's rule to solve for $(\frac{dc^*}{d\theta}, \frac{dm^*}{d\theta}, \frac{dk^*}{d\theta}, \frac{d\sigma^*}{d\theta})$, one gets;

(47)

$$\frac{dc^*}{d\theta} = \frac{\begin{vmatrix} 0 & U_{cm}(\delta-\delta'U)-\delta''UU_cU_m & -\sigma y'' & -\delta \\ -1 & -J_1/U_c & y'' & 0 \\ 0 & 0 & \delta & 0 \\ 0 & \delta'U_m & -y'' & 0 \end{vmatrix}}{|F|}$$

$$= \frac{(\delta)\begin{vmatrix} -1 & -J_1/U_c & y'' \\ 0 & 0 & \delta \\ 0 & \delta'U_m & -y'' \end{vmatrix}}{|F|} = \frac{(\delta)(-1)\begin{vmatrix} 0 & \delta \\ \delta'U_m & -y'' \end{vmatrix}}{|F|} = \frac{\delta^2\delta'U_m}{|F|} > 0$$

(48)

$$\frac{dm^*}{d\theta} = \frac{\begin{vmatrix} U_{cc}(\delta-\delta'U)-\delta''UU_c^2 & 0 & -\sigma y'' & -\delta \\ J_2/U_c & -1 & y'' & 0 \\ -1 & 0 & \delta & 0 \\ \delta'U_c & 0 & -y'' & 0 \end{vmatrix}}{|F|}$$

$$= \frac{(\delta)\begin{vmatrix} J_2/U_c & -1 & y'' \\ -1 & 0 & \delta \\ \delta'U_c & 0 & -y'' \end{vmatrix}}{|F|} = \frac{(\delta)\begin{vmatrix} -1 & \delta \\ \delta'U_c & -y'' \end{vmatrix}}{|F|} = \frac{\delta(y''-\delta\delta'U_c)}{|F|} < 0$$

$$(49) \quad \frac{dk^*}{d\theta} = \frac{\begin{vmatrix} U_{cc}(\delta-\delta'U)-\delta''UU_c^2 & U_{cm}(\delta-\delta'U)-\delta''UU_cU_m & 0 & -\delta \\ J_2/U_c & -J_1/U_c & -1 & 0 \\ -1 & 0 & 0 & 0 \\ \delta'U_c & \delta'U_m & 0 & 0 \end{vmatrix}}{|F|}$$

$$= \frac{\delta\begin{vmatrix} J_2/U_c & -J_1/U_c & -1 \\ -1 & 0 & 0 \\ \delta'U_c & \delta'U_m & 0 \end{vmatrix}}{|F|} = \frac{-\delta\begin{vmatrix} -1 & 0 \\ \delta'U_c & \delta'U_m \end{vmatrix}}{|F|} = \frac{\delta\delta'U_m}{|F|} > 0$$

and finally,

$$(50) \quad \frac{d\sigma^*}{d\theta} = \frac{\begin{vmatrix} U_{cc}(\delta-\delta'U)-\delta''UU_c^2 & U_{cm}(\delta-\delta'U)-\delta''UU_cU_m & -\sigma y'' & 0 \\ J_2/U_c & -J_1/U_c & y'' & -1 \\ -1 & 0 & \delta & 0 \\ \delta'U_c & \delta'U_m & -y'' & 0 \end{vmatrix}}{|F|}$$

$$= \frac{(-1)\begin{vmatrix} U_{cc}(\delta-\delta'U)-\delta''UU_c^2 & U_{cm}(\delta-\delta'U)-\delta''UU_cU_m & -\sigma y'' \\ -1 & 0 & \delta \\ \delta'U_c & \delta'U_m & -y'' \end{vmatrix}}{|F|}$$

$$= \frac{(-1)}{\begin{vmatrix} U_{cm}(\delta-\delta'U)-\delta''UU_c U_m & -\sigma y'' \\ \delta'U_m & -y'' \end{vmatrix}}{|F|}$$

$$+ \frac{\delta \begin{vmatrix} U_{cc}(\delta-\delta'U)-\delta''UU_c^2 & U_{cm}(\delta-\delta'U)-\delta''UU_c U_m \\ \delta'U_c & \delta'U_m \end{vmatrix}}{|F|}$$

$$= \frac{y''[U_{cm}(\delta-\delta'U) - \delta''UU_c U_m] - \sigma y''\delta'U_m + \delta\delta'U_m[U_{cc}(\delta-\delta'U) - \delta''UU_c^2]}{|F|}$$

$$- \frac{\delta\delta'U_c[U_{cm}(\delta-\delta'U) - \delta''UU_c U_m]}{|F|} \gtrless 0$$

Therefore, a change in θ is not neutral in the long-run. The long-run values of consumption and the capital-labor ratio change in the same direction as the change in θ.

Note that if δ were constant, θ would be neutral in the long-run.

3.4: Uzawa's Transformation and Control Problems

Uzawa (1968) introduced the approach used above to endogenize the rate of time preference, applying it to a life-cycle model of consumption behavior. To simplify the problem to a one-state variable one, Uzawa used a transformation of the time scale from t to Δ. In this section, it is shown that for the problem to be transformable it has to be "autonomous" with respect to time. That is, it has to be such that time does not appear explicitly except in the discount factor.

Consider the following control problem (that is similar to ours);

$$(51) \quad \text{Max}_h \int_0^\infty e^{-rt} U(h) \, dt$$

Subject to:

$$(52) \quad \dot{s} = g(h, s, t)$$

and the initial conditions.

Where h is the vector of control variables, s is the state variable and r is the constant rate of time preference.

To endogenize the rate of time preference, define $\Delta(t)$ as in (1) and (2) above. Then the problem now becomes

$$(53) \quad \text{Max}_h \int_0^\infty e^{-\Delta(t)} U(h) \, dt$$

subject to:

$$(52) \quad \dot{s} = g(h, s, t)$$

$$(54) \quad \dot{\Delta} = \delta(U)$$

and the initial conditions.

The transformation would go like this;

from (54), $d\Delta = \delta(U)dt$

and thus,

(55) $dt = \dfrac{d\Delta}{\delta(U)}$

Substituting (55) into (53) and (52), the problem now is a one-state variable one;

(56) $\underset{h}{\text{Max}} \int_0^\infty e^{-\Delta} \dfrac{U(h)}{\delta(U)} d\Delta$

subject to:

(57) $\dfrac{ds}{d\Delta} = \dfrac{g(h,s,t)}{\delta(U)}$

and the initial conditions.

Then the problem is solved, with Δ as the scale variable, as an autonomous problem. The present value Hamiltonian of the problem is,

(58) $H_v = \dfrac{V}{\delta(U)}$

where $V \equiv U(h) + \lambda g(h,s,t)$

The first-order-conditions derived from (58), would look something
like this;

(59) $\quad \dfrac{\partial U}{\partial h} = \dfrac{-\delta\lambda}{\delta-\delta'V} \dfrac{\partial g}{\partial h}$

(60) $\quad \dfrac{d\lambda}{d\Delta} = \dfrac{\lambda\left[\delta - \dfrac{\partial g}{\delta s}\right]}{\delta}$

in addition to the constraints, the initial conditions, and the trans-
versality conditions.

Then the transformation in (55), is used to transform the variables
back into t as the time scale. In particular, equation (60) becomes

(61) $\quad \dot{\lambda} = \lambda\left(\delta - \dfrac{\partial g}{\partial s}\right)$

However, solving the two-state-variable problem directly gives
different first-order-conditions. The Hamiltonian of the problem is

(62) $\quad H = e^{-\Delta(t)}U(h) + \lambda_1\, g(h,s,t) + \lambda_2\, \delta(U)$

The first-order-conditions derived from (62) are:

(63) $\quad \dfrac{\partial U}{\partial h} = \dfrac{-\delta\rho}{\delta+\phi\delta\delta'} \dfrac{\partial g}{\partial h}$

(64) $\quad \dot{\rho} = \rho\left(\delta - \dfrac{\partial g}{\partial s}\right)$

(65) $\quad \dot{\phi} = \delta\phi + U$

in addition to the constraints, the initial conditions, and the trans-
versality conditions.

where $\qquad \rho \equiv e^{\Delta} \lambda_1$

and $\qquad \phi \equiv e^{\Delta} \lambda_2$

For (59) and (63) to be equivalent, $(\delta\phi)$ must equal $(-V)$ for all t.
Assume they are, and totally differentiate with respect to time:

(66) $\quad \delta\dot{\phi} = - \left(U + \lambda g(h,s,t)\right)$

implies that

$$\phi\delta'\frac{\partial U}{\partial h}(\dot{h}) + \delta\dot{\phi} = - \left[\frac{\partial U}{\partial h}\dot{h} + \dot{\lambda}g(h,s,t) + \lambda\left(\frac{\partial g}{\partial h}\dot{h} + \frac{\partial g}{\partial s}\dot{s} + \frac{\partial g}{\partial t}\right)\right]$$

substituting from (61), (65), and (66),

$$\phi\delta'\frac{\partial U}{\partial h}\dot{h} + \delta[-\lambda g(h,s,t)] = - \left[\frac{\partial U}{\partial h}\dot{h} + \lambda\left(\delta - \frac{\partial g}{\partial s}\right)g(h,s,t) + \lambda\frac{\partial g}{\partial h}\dot{h}\right.$$

$$\left. + \lambda\frac{\partial g}{\partial s}g(h,s,t) + \lambda\frac{\partial g}{\partial t}\right]$$

Grouping terms,

(67) $\quad \dot{h}\left[\frac{\partial U}{\partial h}(\phi\delta' + 1) + \lambda\frac{\partial g}{\partial h}\right] = -\lambda\frac{\partial g}{\partial t}$

and since it is assumed that $\delta\phi = -V$:

$$\phi\delta' = \frac{-V\delta'}{\delta}$$

Substituting in (67) gives,

(68) $\dot{h}[\frac{\partial U}{\partial h} (1 - \frac{V\delta'}{\delta}) + \lambda \frac{\partial g}{\partial h}] = -\lambda\frac{\partial g}{\partial t}$

But from (59), the term on the left-hand-side of (68) is zero. Therefore, for (59) and (63) to be equivalent,

$$\frac{\partial g}{\partial t} = 0.$$

That is, the problem has to be autonomous.

If the problem is not autonomous, then the transformation is inapplicable. This is so because the function $\Delta(t)$ depends on the choice variable (h), and thus the correspondence between $\Delta(t)$ and (t) in (1) is not unique. Thus, if one of the exogenous variables is functionally dependent on t, then applying the transformation would result in the choices influencing the transformed value of that variable. Therefore, in this case the correct transformation would transform the problem from one with two-state-variables Δ and s into one with two-state-variables t and s.

Uzawa (1968) and Nairy (1984) deal with autonomous models and thus the transformation is applicable. However, Obstfeld (1981a) and (1981b)

dealt with models that are, like ours, not autonomous. Obstfeld's constraints contain something similar to $\pi(t)$ and $v(t)$ in our model. Thus, his use of the transformation is not valid. However, it should be noted that the two sets of first-order conditions are equivalent at the steady state. This is so because at the steady state $(\dot{\phi})$ in equation (65) is zero and thus

$$\delta\phi = -U \quad \text{(at the steady state)}$$

and from the definition of V, the steady state value of V is equal to U, and thus (66) holds at the steady state.

Therefore, steady-state analysis (like comparative statics) is not affected by the transformation. However, any analysis concerning out-of steady state behavior (like comparative dynamics and stability) is not the same in the two problems.

4:   COMPARATIVE DYNAMICS ANALYSIS

An increase in θ has two kinds of effects on the monetary and real sectors of the economy. First, there is the effect on the steady state values, which has been called the long-run effect, and was determined for the models in Chapters 2 and 3. On the other hand, an increase in θ disturbs the system (in both models), and moves it to a new optimal path. Comparative dynamics analysis provides two kinds of comparisons between the two paths. The "impact" effects compare the values of consumption and real money balances along the two optimal paths for a given capital-labor ratio. The second comparison provided by comparative dynamics analysis is that between the variables along the two optimal paths at each instant of time.

Comparative dynamics techniques for general functional forms are available for two-differential-equations systems, where graphical illustration is of great help. Nagatani's (1981) illustration of Oniki's (1973) concept of comparative dynamics is an example of such techniques. For the three-differential-equations system of Chapter 3, specific functional forms are used when applying comparative dynamics analysis. Fischer (1979) analyzes the impact effects of an increase in θ using a Cobb-Douglas utility function in the model of Chapter 2.

Due to the complexity of four-differential-equations systems, only a discussion of how comparative dynamics analysis would be applied to the model is feasible. This is done in Section 4.1. Section 4.2 summarizes

Fischer's impact effects and completes his analysis by discussing the comparisons at each instant of time.

4.1: Comparative Dynamics

The approach used by Fischer (1979) entails finding the effect of the increase in θ on the negative root of the linearized system around the steady state; then by finding the local (around steady state) approximation of the time path of the capital-labor ratio, the corresponding approximate time paths of the rate of capital accumulation, consumption, and real money balances are derived. Then the effect of an increase in θ on these time paths is analyzed.

To find the effect of an increase in θ on the negative root, the characteristic equation of the linearized system has to be derived. Recall that the linearized system of the model of Chapter 3 is given by (Chapter 3, equation 45)

$$
(1) \quad
\begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \\ \dot{\sigma} \end{bmatrix}
=
\begin{bmatrix}
z_1 & z_2 & z_3 & -\dfrac{\delta}{G} \\[2mm]
\dfrac{mJ_2}{U_c} & \dfrac{-mJ_1}{U_c} & my'' & 0 \\[2mm]
-1 & 0 & \delta & 0 \\[2mm]
\sigma\delta'U_c & \sigma\delta'U_m & -\sigma y'' & 0
\end{bmatrix}
\begin{bmatrix} c-c^* \\ m-m^* \\ k-k^* \\ \sigma-\sigma^* \end{bmatrix}
$$

where $\quad J_1 \equiv U_{mm} - U_{cm}\dfrac{U_m}{U_c}$

$$J_2 \equiv U_{cc}\frac{U_m}{U_c} - U_{cm}$$

$$Z_1 \equiv \frac{1}{G}\{\frac{xmJ_2}{U_c} + U_{cc}(\delta-\delta'U) - \delta''UU_c^2\}$$ (Chapter 3; equation (39))

$$Z_2 \equiv \frac{1}{G}\{\frac{-xmJ_1}{U_c} + U_{cm}(\delta-\delta'U) - \delta''UU_cU_m\}$$ (Chapter 3; equation (40))

$$Z_3 \equiv \frac{1}{G}\{xmy'' - \sigma y''\}$$ (Chapter 3; equation (41))

$$G \equiv \frac{\sigma U_{cc}}{U_c} + \frac{(\sigma-U_c)}{\delta'}\delta''U_c$$ (Chapter 3; equation (31))

and

$$x \equiv -[\frac{\sigma U_{cm}}{U_c} + \frac{(\sigma-U_c)}{\delta'}\delta''U_m]$$ (Chapter 3; equation (32))

The characteristic equation of the system is given by,

(2)

$$\begin{vmatrix} Z_1 - \eta & Z_2 & Z_3 & -\frac{\delta}{G} \\ \frac{mJ_2}{U_c} & \frac{-mJ_1}{U_c} - \eta & my'' & 0 \\ -1 & 0 & \delta-\eta & 0 \\ \sigma\delta'U_c & \sigma\delta'U_m & -\sigma y'' & -\eta \end{vmatrix} = 0$$

where (η) denotes the roots (eigen values) of the system, and all the functions in (2) are evaluated at the steady state.

Expanding (2) gives a fourth degree polynominal relating (η) and θ;

$$(3) \qquad \psi(\eta, \theta) = - \eta^4$$

$$+ \eta^3 \left[Z_1 - \frac{mJ_1}{U_c} + \delta \right]$$

$$+ \eta^2 \left[\frac{\delta\sigma\delta'U_c}{G} + Z_3 + \delta\left(\frac{mJ_1}{U_c} - Z_1\right) + \frac{Z_1 mJ_1}{U_c} + \frac{Z_2 mJ_2}{U_c} \right]$$

$$+ \eta \left[\frac{\delta}{G}\left(\sigma y'' + \frac{\sigma\delta'mU_m J_2}{U_c} + \sigma\delta'mJ_1 - \delta\sigma\delta'U_c\right) + Z_2 my'' \right.$$

$$\left. + \frac{Z_3 mJ_1}{U_c} - \frac{\delta Z_1 mJ_1}{U_c} - \frac{\delta Z_2 mJ_2}{U_c} \right]$$

$$+ \frac{\delta}{G}\left[\frac{\sigma mJ_1 y''}{U_c} - \sigma\delta'mU_m y'' - \delta\left(\frac{\sigma m\delta'U_m J_2}{U_c} + \sigma\delta'mJ_1\right) \right] = 0$$

Appendix B proves that (3) has a unique negative root. Note that,

$$(4) \qquad \psi(0, \theta) = \frac{\delta}{G}\left\{ \frac{\sigma mJ_1 y''}{U_c} - \sigma\delta'mU_m y'' - \delta\left(\frac{\sigma m\delta'U_m J_2}{U_c} + \sigma\delta'mJ_1\right) \right\} < 0$$

since it is equal to the determinant of the system.

Since only the negative root is of interest, equations (3) and (4) imply that

$$(5) \qquad \frac{\partial\psi}{\partial\eta} < 0 \quad \text{(at the negative root).}$$

To find the effect of an increase in θ on η, equation (3) is totally differentiated, giving

$$(6) \qquad \frac{d\eta}{d\theta} = \frac{-\partial\psi/\partial\theta}{\partial\psi/\partial\eta}$$

Therefore, equation (5) implies that the sign of $\left(\frac{d\eta}{d\theta}\right)$ is that of $\left(\frac{\partial\psi}{\partial\theta}\right)$. Finding $\left(\frac{\partial\psi}{\partial\theta}\right)$ requires, among other things, knowing third derivatives of the utility function, the rate of time preference function, and the production function. That is why specific functional forms are used in the literature.

The local approximation of the time path of the capital-labor ratio is given by, (where * denotes steady state values)

$$(7) \qquad k_t = k* - \left(k* - k_0\right) e^{\eta t}$$

where $\eta < 0$.

From equation (2) (third row), the corresponding approximate path of consumption is given by,

$$(8) \qquad c_t = c* + \left(\delta - \eta\right)\left(k_t - k*\right)$$

and that of real money balances is (from equation (2); second row; and using (8)),

$$(9) \quad m_t = m^* + \frac{\dfrac{mJ_2}{U_c} + my''}{\dfrac{mJ_1}{U_c} + \eta} \left(k_t - k^*\right)$$

and differentiating (7) with respect to time, gives the path for the rate of capital accumulation,

$$(10) \quad \dot{k} = -\eta \left(k^* - k_t\right)$$

Then given $\dfrac{d\eta}{d\theta}$, $\dfrac{dc^*}{d\theta}$, $\dfrac{dm^*}{d\theta}$, and $\dfrac{dk^*}{d\theta}$, equations (8)-(10) are differentiated with respect to θ holding k_t constant to come up with the impact effects. For the comparisons at each instant of time, $\dfrac{dk_t}{d\theta}$ is first found from equation (7) and then equations (8)-(10) are differentiated with respect to θ.

4.2: Fischer's Results for the Model of Chapter 2

The specific utility function used by Fischer is a Cobb-Douglas of the form,

$$U = \frac{\left(c^\alpha m^\beta\right)^{1-R}}{1-R}$$

where α, β, $R > 0$, $\alpha + \beta < 1$ and $R \neq 1$.

Fischer shows that the negative root of the system (η) decreases (becomes larger in absolute value) as θ is increased. That is,

(11) $\dfrac{dn}{d\theta} < 0$

The approximate time paths for Fischer's example of the constant rate of time preference model that correspond to (7)-(10) are (Fischer, 1979, p. 1438) given by (7), (8), (10) and

(12) $m_t = m* + \dfrac{\beta}{\alpha} \dfrac{\delta - \eta - \dfrac{y''c*}{\delta + \theta}}{\delta + \theta - \eta} (k_t - k*)$

Recalling that $\dfrac{dc*}{d\theta} = \dfrac{dk*}{d\theta} = 0$, and $\dfrac{dm*}{d\theta} < 0$, equation (9), (10), and (12) are differentiated for a given $k_t < k*$ to come up with the _impact_ effects of an increase in θ on $\overset{\bullet}{k}_t$, c_t, and m_t; from equation (10),

(13) $\dfrac{d\overset{\bullet}{k}_t}{d\theta} = - (k* - k_t) \dfrac{dn}{d\theta} > 0$

Therefore, for a given $k_t < k*$, the rate of capital accumulation increases with the increase in θ.

From equation (8),

(14) $\dfrac{dc_t}{d\theta} = - (k_t - k*) \dfrac{dn}{d\theta} < 0$

Therefore, for a given $k_t < k*$, consumption is smaller with the larger θ.

Finally, from equation (12),

$$(15) \quad \frac{dm_t}{d\theta} = \frac{dm*}{d\theta} + \frac{\beta}{\alpha} \left(k_t - k*\right) \left\{ \frac{-\frac{dn}{d\theta} + \frac{y''c*}{(\delta+\theta)^2}}{\delta + \theta - n} - \frac{\left(\delta - n - \frac{y''c*}{\delta+\theta}\right)\left(1 - \frac{dn}{d\theta}\right)}{(\delta + \theta - n)^2} \right\}$$

Thus, the impact effect on real money balances is ambiguous, which, as noted by Fischer (1979, p. 1438), is contrary to what one would expect. In the model at hand, one would guess that, for a given $k_t < k*$, real money balances will be smaller along the new optimal path.

To complete Fischer's analysis, we compare the values of k_t, $\dot{k}_t$, c_t, and m_t along the new optimal path (after the increase in θ) with those along the old optimal path (before the increase in θ) at each point in time. Given the local approximations of the time path of the variables in (7)-(10), and (12), the comparison at each point in time is achieved by differentiating the equations with respect to θ, assuming $k_0 \leq k_t \leq k*$.

From equation (7),

$$(16) \quad \frac{dk_t}{d\theta} = -t \frac{dn}{d\theta} \left(k* - k_o\right) e^{nt} > 0 \quad \forall t \quad (= 0 \text{ at } t=0)$$

Thus, except for the initial capital-labor ratio, the increase in θ results in a larger capital-labor ratio along the new optimal path at all instants of time.

From equation (10)

$$(17) \quad \frac{d\dot{k}_t}{d\theta} = -\frac{dn}{d\theta} \left(k* - k_t\right) + n \frac{dk_t}{d\theta} \gtrless 0$$

Thus, in general it is not clear how the rate of capital accumulation along the new optimal path compares with that along the old path. However, since k_0 is unchanged, then from the impact effects, the rate of capital accumulation is larger initially. Equation (17) suggests that after some point in time the rate of capital accumulation becomes smaller along the new path, which actually must be the case for k_t to converge to k^*.

Differentiating (8) with respect to θ,

$$(18) \quad \frac{dc_t}{d\theta} = - \frac{dn}{d\theta} \left(k_t - k^* \right) + \left(\delta - n \right) \frac{dk_t}{d\theta} = \delta \frac{dk_t}{d\theta} - \frac{d\dot{k}_t}{d\theta} \gtrless 0$$

However, since k_0 is constant, the impact effects imply that consumption along the new path is smaller initially. Equation (18) shows that when the rate of capital accumulation becomes smaller, consumption becomes larger along the new path.

Finally, differentiating (12) with respect to θ,

$$(19) \quad \frac{dm_t}{d\theta} = \frac{dm^*}{d\theta} + \frac{\beta}{\alpha} \left(k_t - k^* \right) \left\{ \frac{- \frac{dn}{d\theta} + \frac{y'' c^*}{(\delta+\theta)^2}}{\delta + \theta - n} - \frac{\left(\delta - n - \frac{y'' c^*}{\delta+\theta} \right) \left(1 - \frac{dn}{d\theta} \right)}{\left(\delta + \theta - n \right)^2} \right\}$$

$$+ \frac{\beta}{\alpha} \left[\frac{\delta - n - \frac{y'' c^*}{\delta+\theta}}{\delta + \theta - n} \right] \frac{dk_t}{d\theta} \gtrless 0$$

Therefore, in general it is not clear how real money balances along the new path compare with those along the old one.

5: SUMMARY OF RESULTS AND CONCLUSIONS

Section 5.1 summarizes the results of the analysis, presenting a comparison between the models of Chapters 2 and 3. It also provides some intuition for the various results. In section 5.2, some suggestions for further research are offered.

5.1: Summary and Intuition

In Chapter 2, we derived Sidrauski's (1967b) equations and proved his results. We also derived Fischer's (1979) system in c, m, and k, and formally analyzed the stability of the model and proved the long-run neutrality of money.

We have found that in the monetary growth model of Sidrauski (1967b) or that of Fischer (1979), the assumption of a constant rate of time preference generates the characteristic of long-run money neutrality. In such models, the steady state (long-run) capital-labor ratio is such that its marginal product is the sum of the rate of time preference (δ), the growth rate of the population (n), and the rate of capital depreciation (μ). An increase in the growth rate of nominal money balances (θ) has no effect on δ, n, or μ, leaving the steady state capital-labor ratio unchanged. Therefore, changes in the growth rate of money have no effect on the real sector of the economy. The only long-run effect of an increase in θ is to reduce the steady state value of real money balances. The increase in θ increases the steady state value of the expected inflation rate by the same amount, reducing the real return on holding money.

This reduction in the real rate of return, induces the agents in the economy to hold less of the asset, money. Since the steady state values of the capital-labor ratio and consumption stay the same, the representative individual, and thus the economy as a whole, is worse off due to the increase in the growth rate of nominal money balances. They achieve a lower level of utility at the new steady state.

An increase in θ moves the economy to a new steady state. During the transition period, the economy is operating along a new optimal saddle path. The impact effects derived by Fischer (1979) for a specific utility function show that for a given capital stock, the rate of capital accumulation is larger and consumption is smaller after the increase in θ. The effect on real money balances is not clear; they might be larger or smaller. However, due to the increase in the inflation rate one would expect real money balances to be smaller after the increase in θ.

Comparative dynamic analysis, which compares the values of the variables along the new optimal path with those along the old optimal path, shows that for Fischer's example, the capital stock is larger along the new optimal path at all points in time (except for the initial value of the capital stock, which is the same for both paths). In general, the behaviors of the rate of capital accumulation and consumption are ambiguous. However, a likely behavior is that the rate of capital accumulation rises and consumption falls during the early part of the transition period; and as the new path approaches the new steady state, the rate of capital accumulation becomes smaller and consumption becomes larger than their values along the old optimal path for the same points

in time. This seems to make sense; the increase in θ causes the inflation rate to increase lowering the real rate of return on money, inducing a shift to holding more capital (in the short run) but reducing consumption. However, the added capital increases disposable income and given enough time (the new steady state is approached) the income effect increases consumption.

The analysis does not give definite conclusions on how real money balances along the new optimal path compare with those along the old one.

With relation to stability, the model of Chapter 2 was found to have the saddle-point stability property, which implies that among the infinitely many paths, for each initial capital-labor ratio there is only one stable path leading to the steady state. This path is usually termed the stable "arm" or "manifold" of the saddle point (Burmeister 1980). Therefore, the economy can reach the saddle point only if it starts on the stable arm. Thus, for any initial capital-labor ratio, there has to exist a mechanism to ensure the "right" choice of the initial values of the other variables so that the system starts at a point on the stable path. In infinite horizon optimal control problems, the transversality condition assures the choice of the "right" initial values. The transversality condition is essentially equivalent to assuming that markets always clear at finite positive prices. Starting at any point off the stable path leads to markets not clearing, and thus prices are driven to either zero or infinity at some finite future instance of time.

In Chapter 3, the rate of time preferences was endogenized using Uzawa's (1968) approach. The approach was chosen over others for its relative simplicity and because it maintains the basic structure of the Sidrauski model as explained in Chapter 1. The endogenization of the rate of time preference altered the long-run effects of an increase in θ, while the short-run effects are similar to those of the constant rate of time preference model.

In the long-run, an increase in θ increases the capital-labor ratio and consumption while reducing real money balances. The capital-labor ratio is still such that its marginal product is the sum of the rate of time preference (δ), the growth rate of the population (n), and the rate of capital depreciation (μ). However, δ in this case is a function of utility and thus of consumption and real money balances. In this model, δ provides a link between the monetary sector and the real sector. At the new steady state, the smaller real money balances reduce utility and thus the rate of time preference. For the steady state relation between δ and the capital-labor ratio to hold at the new steady state, the marginal product of capital has to fall. Therefore, the new steady state capital-labor ratio is larger and thus so is consumption.

The increase in θ increases the steady state value of the expected rate of inflation, reducing the real return of holding money. This induces individuals to reduce their holdings of real money balances. However, instead of real money balances accounting for the full adjustment, as in the constant rate of time preference model, the portfolio adjustment also involves the long-run capital-labor ratio. As real money

balances fall the rate of time preference becomes smaller than the real
interest rate, providing an incentive for agents to hold more capital and
thus allowing for higher consumption at the new steady state.

In comparison with the steady state before the increase in θ, the
new steady state corresponds to a lower level of utility. This can be
seen by comparing the rates of time preference. The new steady state
corresponds to a lower rate of time preference; from the assumptions on
δ, it must correspond to a smaller utility level. Therefore, an increase
in θ makes the economy worse-off, as it did in the model with a constant
rate of time preference. However, the fall in steady state utility is
smaller for the case where the rate of time preference is variable. To
show that, consider the steady state utility (where * denotes steady
state);

(1) $U^* = U\left(c^*, m^*\right)$

which by differentiation with respect to θ gives,

(2) $\dfrac{dU^*}{d\theta} = U_c \dfrac{dc^*}{d\theta} + U_m \dfrac{dm^*}{d\theta}$

For the constant rate of time preference model of Chapter 2, $\dfrac{dc^*}{d\theta} = 0$, and

(3) $\dfrac{dm^*}{d\theta} = \dfrac{U_c}{J_1}$ (Chapter 2, equation (95))

And thus for the model of Chapter 2,

$$(4) \qquad \frac{dU^*}{d\theta} = \frac{U_m U_c}{J_1}$$

On the other hand, for the variable rate of time preference model of Chapter 3,

$$(5) \qquad \frac{dc^*}{d\theta} = \frac{\delta^2 \delta' U_m}{|F|} \qquad \text{(Chapter 3, equation (47))}$$

and

$$(6) \qquad \frac{dm^*}{d\theta} = \frac{\delta(y'' - \delta \delta' U_c)}{|F|} \qquad \text{(Chapter 3, equation (48))}$$

Therefore, for the model of Chapter 3,

$$(7) \qquad \frac{dU^*}{d\theta} = U_c \left(\frac{\delta^2 \delta' U_m}{F} \right) + \frac{U_m \delta y''}{|F|} - \frac{\delta^2 \delta' U_c U_m}{|F|} = \frac{U_m \delta y''}{|F|}$$

where $|F| = \delta \left\{ \dfrac{J_1 y''}{U_c} - \delta' U_m y'' - \delta \left(\dfrac{\delta' J_2 U_m}{U_c} + \delta' J_1 \right) \right\}$

Let $S \equiv -\delta' U_m y'' - \delta \left(\dfrac{\delta' J_2 U_m}{U_c} + \delta' J_1 \right) > 0$

Therefore,

$$(8) \qquad |F| = \delta \left[\frac{J_1 y''}{U_c} + S \right]$$

Using (8), equation (7) becomes,

$$(9) \qquad \frac{du^*}{d\theta} = \frac{U_m y''}{\left(\dfrac{J_1 y''}{U_c} + S\right)}$$

To compare (4) and (9), rewrite equation (4) as

$$(10) \qquad \frac{du^*}{d\theta} = \frac{U_m y''}{\left(\dfrac{J_1 y''}{U_c}\right)} \qquad \text{(for the model of Chapter 2)}$$

Since $y'' < 0$, and $\dfrac{J_1 y''}{U_c}$ and $S > 0$, then

$$\frac{dU^*}{d\theta} \text{ (in (10))} < \frac{dU^*}{d\theta} \text{ (in (9))}$$

Therefore, the fall in steady state utility is smaller in the model of Chapter 3. However, the economy with the variable rate of time preference will have a lower rate of time preference and thus a lower real interest rate.

Therefore, endogenizing the rate of time preference using Uzawa's approach gives the expected results for the steady state portfolio adjustment when the growth rate of money is increased. The change in the

relative real rates of return that results from the increase in θ induces the shift to the asset with the relatively higher real return. Thus, the structure here implies that long-run asset demands are functions of relative rates of return as well as own rates, an implication that might be more acceptable than that of the model with the constant rate of time preference. The long-run neutrality of money in that model essentially implies that long-run asset demands are a function only of own rates of return.

However, given the structure of the model at hand, endogenizing the rate of time preference the way we did implies that the economy with the high capital stock will have a lower real interest rate and a lower rate of time preference.

As far as stability is concerned, the steady state of the model with the variable rate of time preference has the saddle point property.

5.2: Suggestions for Further Research

The way the rate of time preference was endogenized needs further exploration. A study of the axiomatic basis for the Uzawa (1968) approach is needed. Uzawa (1968) states some axioms which he claims, imply the existence of the function $\delta(U)$. However, they seem to be too restrictive; they assume "too much" about the utility function. A study of the conditions under which preferences will display variable rates of time preference would be along the lines of the Koopmans, Diamond, and Williamson (1964) paper.

Another obvious extension is to models which are both explicitly stochastic and incorporate heterogeneity in either households or capital goods, say along the lines of Epstein (1983) and Becker (1980).

Finally, the short-run effects need more exploration. The effects we found in this study are comparisons between values along optimal paths. However, a study of the behavior of the system when it is off the optimal path is needed, particularly since the model displays the saddle-point property. As explained above, a mechanism is needed to assure that when the system is disturbed it moves to a new optimal path. This is usually achieved by assuming the problem away through assuming market clearing at all times, transversality conditions (boundedness), convergence of expectations or allowing for discontinuous jumps in the price level, etc. See for example Burmeister (1980) and Sargent and Wallace (1973). What is needed is a technique that explains the behavior of the system when a disturbance causes a movement off the optimal path.

6: BIBLIOGRAPHY

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7: APPENDIX A

In this Appendix, a different approach (than that used in the text; Chapter 2) is used to derive the stability condition for the individual optimization problem in Sidrauski (1967b).

This approach eliminates k from the first-order-condtions ((13) and (14)), using the stock constraint and condition (16). Then comparative statics is applied to the resulting system to derive the partials of c and m with respect to λ, π, and a in demand functions (18) and (19). Then the stability of the system is analyzed.

Therefore, the system to which comparative statics is to be applied is;

(A1) $\quad U_c(c, m) = \lambda$

(A2) $\quad U_m(c, m) = \lambda[\pi + y'(a - m) - \mu]$

Totally differentiating (A1) and (A2) gives;

(A3) $\quad U_{cc}dc + U_{cm}dm = d\lambda$

(A4) $\quad U_{mc}dc + U_{mm}dm = d\lambda[\pi + y' - \mu] + \lambda d\pi - \lambda y''dm + \lambda y''da$

Rewriting (A4);

(A5) $\quad U_{mc}dc + (U_{mm} + \lambda y'')dm = d\lambda[\pi + y' - \mu] + \lambda d\pi + \lambda y''da$

To find the partials of c and m with respect to λ, set $d\pi = da = 0$ in (A5) and solve the resulting system for $dc/d\lambda$ and $dm/d\lambda$.

In matrix form, the solution is

$$(A6) \quad \begin{bmatrix} dc/d\lambda \\ dm/d\lambda \end{bmatrix} = \begin{bmatrix} U_{cc} & U_{cm} \\ U_{mc} & U_{mm} + \lambda y'' \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ \pi + y' - \mu \end{bmatrix}$$

$$= \frac{1}{U_{cc}U_{mm} + \lambda U_{cc}y'' - U_{cm}^2} \begin{bmatrix} U_{mm} + \lambda y'' & -U_{cm} \\ -U_{cm} & U_{cc} \end{bmatrix} \begin{bmatrix} 1 \\ \pi + y' - \mu \end{bmatrix}$$

$$= \frac{1}{J + \lambda U_{cc}y''} \begin{bmatrix} U_{mm} + \lambda y'' & -U_{cm} \\ -U_{cm} & U_{cc} \end{bmatrix} \begin{bmatrix} 1 \\ \pi + y' - \mu \end{bmatrix}$$

Therefore,

$$\frac{dc}{d\lambda} = \frac{U_{mm} + \lambda y'' - U_{cm}(\pi + y' - \mu)}{J + \lambda U_{cc}y''}$$

But from the first order conditions

$$(A7) \quad \pi + y' - \mu = \frac{U_m}{U_c}$$

Thus,

$$(A8) \quad \frac{dc}{d\lambda} = \frac{J_1 + \lambda y''}{J + \lambda U_{cc}y''}$$

Similarly,

$$(A9) \quad \frac{dm}{d\lambda} = \frac{-U_{cm} + U_{cc}(\pi + y' - \mu)}{J + \lambda U_{cc}y''} = \frac{J_2}{J + \lambda U_{cc}y''}$$

To find the partials of c and m with respect to π, set $d\lambda = da = 0$ in (A3) and (A5) and solve the resulting system for $dc/d\pi$ and $dm/d\pi$:

$$(A10) \quad \begin{bmatrix} dc/d\pi \\ dm/d\pi \end{bmatrix} = \frac{1}{J + \lambda U_{cc} y''} \begin{bmatrix} U_{mm} + \lambda y'' & -U_{cm} \\ -U_{cm} & U_{cc} \end{bmatrix} \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$$

Therefore,

$$(A11) \quad \frac{dc}{d\pi} = \frac{-U_{cm}\lambda}{J + \lambda U_{cc} y''}$$

and

$$(A12) \quad \frac{dm}{d\pi} = \frac{U_{cc}\lambda}{J + \lambda U_{cc} y''}$$

Finally, the partials of c and m with respect to a are found the same way. Setting $d\lambda = d\pi = 0$ in (A3) and (A5) and solving the resulting system:

$$(A13) \quad \begin{bmatrix} dc/da \\ dm/da \end{bmatrix} = \frac{1}{J + \lambda U_{cc} y''} \begin{bmatrix} U_{mm} + \lambda y'' & -U_{cm} \\ -U_{cm} & U_{cc} \end{bmatrix} \begin{bmatrix} 0 \\ \lambda y'' \end{bmatrix}$$

Therefore,

$$(A14) \quad \frac{dc}{da} = \frac{-U_{cm}\lambda y''}{J + \lambda U_{cc} y''}$$

and

$$(A15) \quad \frac{dm}{da} = \frac{U_{cc}\lambda y''}{J + \lambda U_{cc}y''}$$

The system for the individual's optimization problem is described by the two differential equations:

$$(A16) \quad \dot{\lambda} = \lambda \cdot \left(\delta - y'(a-m) + \mu + n\right)$$

and

$$(A17) \quad \dot{a} = y(a-m) + v - (\pi+n)m - (\mu+n)(a-m) - c$$

where c and m are functions of λ, π, and a.

To analyze the stability of the system, linearize (A16) and (A17) around the steady state values of λ and a.

Linearizing (A16) gives: (* denotes steady state values)

$$(A18) \quad \dot{\lambda} = \lambda * y'' \frac{\partial m}{\partial \lambda}(\lambda - \lambda *) - \lambda *\left(y'' - y'' \frac{\partial m}{\partial a}\right)(a - a*)$$

Similarly (A17) gives:

$$\dot{a} = \left[-y'\frac{\partial m}{\partial \lambda} - (\pi+n)\frac{\partial m}{\partial \lambda} + (\mu+n)\frac{\partial m}{\partial \lambda} - \frac{\partial c}{\partial \lambda}\right](\lambda - \lambda *)$$

$$+ \left[y' - y'\frac{\partial m}{\partial a} - (\pi+n)\frac{\partial m}{\partial a} - (\mu+n) + (\mu+n)\frac{\partial m}{\partial a} - \frac{\partial c}{\partial a}\right](a - a*)$$

$$= \left[(-y' - \pi + \mu)\frac{\partial m}{\partial \lambda} - \frac{\partial c}{\partial \lambda}\right](\lambda - \lambda *)$$

$$+ \left[(y' - \mu+n)) + (-y' - \pi + \mu)\frac{\partial m}{\partial a} - \frac{\partial c}{\partial a}\right](a - a*)$$

and using (A7),

$$(A19) \quad \overset{\bullet}{a} = \left[\frac{-U_m}{U_c} \frac{\partial m}{\partial \lambda} - \frac{\partial c}{\partial \lambda} \right] (\lambda - \lambda^*) + \left[(y' - (\mu+n)) - \frac{U_m}{U_c} \frac{\partial m}{\partial a} - \frac{\partial c}{\partial a} \right] (a - a^*)$$

Putting (A18) and (A19) in matrix form;

$$(A20) \quad \begin{bmatrix} \overset{\bullet}{\lambda} \\ \\ \overset{\bullet}{a} \end{bmatrix} = \begin{bmatrix} \lambda y'' \frac{\partial m}{\partial \lambda} & -\lambda y''\left(1 - \frac{\partial m}{\partial a}\right) \\ \\ \frac{-U_m}{U_c} \frac{\partial m}{\partial \lambda} - \frac{\partial c}{\partial \lambda} & (y' - (\mu+n)) - \frac{U_m}{U_c} \frac{\partial m}{\partial a} - \frac{\partial c}{\partial a} \end{bmatrix} \begin{bmatrix} \lambda - \lambda^* \\ \\ a - a^* \end{bmatrix}$$

The determinant of the matrix in (A20) is;

$$(A21) \quad \lambda y'' \frac{\partial m}{\partial \lambda} (y' - (\mu+n)) - \lambda y'' \frac{\partial m}{\partial \lambda} \frac{U_m}{U_c} \frac{\partial m}{\partial a} - \lambda y'' \frac{\partial m}{\partial \lambda} \frac{\partial c}{\partial a}$$

$$- \lambda y'' \frac{U_m}{U_c} \frac{\partial m}{\partial \lambda} - \lambda y'' \frac{\partial c}{\partial \lambda} + \lambda y'' \frac{\partial m}{\partial a} \frac{\partial m}{\partial \lambda} \frac{U_m}{U_c} + \lambda y'' \frac{\partial m}{\partial a} \frac{\partial c}{\partial \lambda}$$

$$= \lambda y'' \frac{\partial m}{\partial \lambda} (y' - (\mu+n)) - \lambda y'' \frac{\partial m}{\partial \lambda} \frac{\partial c}{\partial a} - \lambda y'' \frac{U_m}{U_c} \frac{\partial m}{\partial \lambda} - \lambda y'' \frac{\partial c}{\partial \lambda} + \lambda y'' \frac{\partial m}{\partial a} \frac{\partial c}{\partial \lambda}$$

From equations (A8), (A9), (A14), and (A15),

$$(A22) \quad \frac{\partial m}{\partial \lambda} \frac{\partial c}{\partial a} = \frac{J_2}{J + \lambda U_{cc} y''} \left(\frac{-U_{cm} \lambda y''}{J + \lambda U_{cc} y''} \right) = \frac{-J_2 U_{cm} \lambda y''}{[J + \lambda U_{cc} y'']^2}$$

and

(A23) $\quad \dfrac{\partial m}{\partial a} \cdot \dfrac{\partial c}{\partial \lambda} = \dfrac{U_{cc}\lambda y''(J_1 + \lambda y'')}{\left[J + \lambda U_{cc}y''\right]^2} = \dfrac{U_{cc}\lambda y'' J_1}{\left[J + \lambda U_{cc}y''\right]^2} + \dfrac{U_{cc}(\lambda y'')^2}{\left[J + \lambda U_{cc}y''\right]^2}$

and from the definition J_1;

$$U_{cc} J_1 = U_{cc}\left[U_{mm} - U_{cm}\dfrac{U_m}{U_c}\right]$$

(adding and subtracting U_{cm}^2 on the right-hand side)

(A24) $\quad U_{cc} J_1 = U_{cc} U_{mm} - U_{cm} U_{cc} \dfrac{U_m}{U_c} - U_{cm}^2 + U_{cm}^2 = J - U_{cm} J_2$

Substituting (A24) into (A23) and rearranging;

$$\dfrac{\partial m}{\partial a} \cdot \dfrac{\partial c}{\partial \lambda} = \dfrac{\lambda y'' J}{\left[J + \lambda U_{cc}y''\right]^2} - \dfrac{\lambda y'' J_2 U_{cm}}{\left[J + \lambda U_{cc}y''\right]^2} + \dfrac{U_{cc}(\lambda y'')^2}{\left[J + \lambda U_{cc}y''\right]^2}$$

and taking (A22) into account;

(25) $\quad \dfrac{\partial m}{\partial a} \dfrac{\partial c}{\partial \lambda} = \dfrac{\lambda y'' J}{\left[J + \lambda U_{cc}y''\right]^2} + \dfrac{U_{cc}(\lambda y'')^2}{\left[J + \lambda U_{cc}y''\right]^2} + \dfrac{\partial m}{\partial \lambda} \dfrac{\partial c}{\partial a}$

From (A7),

(A26) $\quad y' - (\mu + n) = \dfrac{U_m}{U_c} - (\pi + n)$

Substituting (A25) and (A26) into (A21), the determinant becomes;

$$(A27) \quad \lambda y'' \frac{\partial m}{\partial \lambda} \left[\frac{U_m}{U_c} - (\pi+n) \right] - \lambda y'' \frac{\partial m}{\partial \lambda} \frac{\partial c}{\partial a} - \lambda y'' \frac{U_m}{U_c} \frac{\partial m}{\partial \lambda} - \lambda y'' \frac{\partial c}{\partial \lambda}$$

$$+ \lambda y'' \left\{ \frac{\lambda y'' J}{\left[J + \lambda U_{cc} y'' \right]^2} + \frac{U_{cc} (\lambda y'')^2}{\left[J + \lambda U_{cc} y'' \right]^2} + \frac{\partial m}{\partial \lambda} \frac{\partial c}{\partial a} \right\}$$

$$= - \lambda y'' \frac{\partial m}{\partial \lambda} (\pi+n) - \lambda y'' \frac{\partial c}{\partial \lambda} + \lambda y'' \left\{ \frac{\lambda y'' J}{\left[J + \lambda U_{cc} y'' \right]^2} + \frac{U_{cc} (\lambda y'')^2}{\left[J + \lambda U_{cc} y'' \right]^2} \right\}$$

$$= - \lambda y'' \left\{ (\pi+n) \frac{\partial m}{\partial \lambda} + \frac{\partial c}{\partial \lambda} - \frac{\lambda y'' J + U_{cc} (\lambda y'')^2}{\left[J + \lambda U_{cc} y'' \right]^2} \right\}$$

Substituting for $\frac{\partial m}{\partial \lambda}$ and $\frac{\partial c}{\partial \lambda}$ in (A27);

The determinant becomes

$$(A28) \quad = -\lambda y'' \left\{ (\pi+n) \frac{J_2}{J + \lambda U_{cc} y''} + \frac{J_1 + \lambda y''}{J + \lambda U_{cc} y''} - \frac{\lambda y'' J + U_{cc} (\lambda y'')^2}{\left[J + \lambda U_{cc} y'' \right]^2} \right\}$$

$$= \frac{-\lambda y''}{J + \lambda U_{cc} y''} \left\{ (\pi+n) J_2 + J_1 + \lambda y'' - \frac{\lambda y'' (J + U_{cc} \lambda y'')}{J + \lambda U_{cc} y''} \right\}$$

$$= \frac{-\lambda y''}{J + \lambda U_{cc} y''} \left\{ (\pi+n) J_2 + J_1 \right\}$$

Since λ and J are positive and U_{cc} and y'' are negative, the sign of the determinant (A28) is that of $\{(\pi+n) J_2 + J_1\}$. Since in a two differential equation system, a sufficient and necessary condition for saddle point stability is that the determinant be negative, the stability condition for the individual's optimization problem of Sidrauski is

(A29) $(\pi+n) J_2 + J_1 < 0$

Condition (A29) is the same condition derived in Chapter 2; condition (38).

8: APPENDIX B:

A PROOF OF THE UNIQUENESS OF THE NEGATIVE ROOT OF

THE LINEARIZED SYSTEM OF CHAPTER 3

In this appendix, we prove that the linearized system of Chapter 3 (equation (45)) has only one negative root. To do that the characteristic equation (Chapter 4, equation (3)) is used.

Rewriting the characteristic equation gives,

$$
\text{(B1)} \quad G\eta^4 - \eta^3\left[2G\delta + m\left(xJ_2 - GJ_1\right)\right]
$$

$$
+ \eta^2\left[G\delta^2 + 2m\delta\left(xJ_2 - GJ_1\right) + \sigma\delta\delta'U_c - y''(\sigma - xm)\right]
$$

$$
+ \eta\left[-\sigma\delta^2\delta'U_c - m\delta^2\left(xJ_2 - GJ_1\right) + \delta y''(\sigma - xm)\right.
$$

$$
\left. + \sigma\delta\delta'm\left(U_cJ_1 + U_mJ_2\right) - mJ_1\sigma y''\right]
$$

$$
- \left(m\delta\sigma\right)\left[\delta\delta'U_cJ_1 + \delta J_2\delta'U_m + \delta'U_my'' - J_1y''\right] = 0
$$

where η denotes the roots of the system.

$$
x \equiv -\left[\frac{U_{cm}\sigma}{U_c} + \frac{(\sigma - U_c)}{\delta'}\delta''U_m\right]
$$

$$
G \equiv \frac{U_{cc}\sigma}{U_c} + \left(\frac{\sigma - U_c}{\delta'}\right)\delta''U_c
$$

$$
J_1 = U_{mm} - U_{cm}\frac{U_m}{U_c} < 0
$$

and

$$J_2 = U_{cc} \frac{U_m}{U_c} - U_{cm} < 0$$

Equation (B1) can be rewritten as;

(B2) $(\eta-\delta) \{ G\eta^3 + \eta^2[-G\delta-m(xJ_2-GJ_1)] + \eta[m\delta(xJ_2-GJ_1) + \sigma\delta\delta'U_c-y''(\sigma-xm)]$

$+ (\sigma\delta\delta'm)(U_cJ_1 + U_mJ_2) - mJ_1\sigma y''\} - m\delta\sigma\delta'U_my'' = 0$

Let

(B3) $\quad F(\eta) \equiv G\eta^3 + b_1\eta^2 + b_2\eta + b_3$

where $\quad b_1 \equiv - G\delta - m(xJ_2 - GJ_1)$

$b_2 \equiv m\delta(xJ_2 - GJ_1) + \sigma\delta\delta'U_c - y''(\sigma-xm)$

and

$b_3 \equiv (\sigma\delta\delta'm)(U_cJ_1 + U_mJ_2) - mJ_1\sigma y''$

Since J_1, J_2, and $y'' < 0$, then $b_3 < 0$.

At the end of this appendix, it is shown that at steady state

$$xJ_2 - GJ_1 < 0$$

Therefore, b_1 is positive and b_2 is ambiguous in sign. But no matter what the sign of b_2 is, there are two sign changes in the coefficients of

$F(\eta)$, implying that $F(\eta)=0$ has one negative root $(\tilde{\eta})$ and two positive (or complex with positive real parts) roots.

Since b_3 is negative, then, for $\eta < 0$, we have

$$\text{(B4)} \qquad F(\eta) \gtrless 0 \text{ as } \eta \lessgtr \tilde{\eta}$$

Equation (B4) also implies that $\frac{dF}{d\eta} < 0$ $\left(\text{for } \eta \leq \tilde{\eta}\right)$

Now, rewriting (B2), using B3, the characteristic equation becomes

$$\text{(B5)} \qquad (\eta-\delta)\,(F(\eta)) + b_4 = 0$$

where $b_4 \equiv - m\delta\sigma\delta' U_m y'' > 0$

Since b_4 is positive, then for (B5) to hold,

$$\text{(B6)} \qquad (\eta - \delta)\, F(\eta) < 0$$

Let η_1 be a negative root of (B5).

When $\eta = \eta_1 < 0$, $(\eta_1 - \delta)$ is negative, and thus equation (B6) implies

$$\text{(B7)} \qquad F(\eta_1) > 0$$

and hence, from (B4), $\eta_1 < \tilde{\eta}$.

Thus, the root of (B5), η_1 is less than $\tilde{\eta}$. Differentiating (B5) with respect to η (and evaluating at $\eta = \eta_1$) gives,

(B8) $\quad \dfrac{d}{d\eta}[(\eta-\delta)F(\eta) + b_4]\Big|_{\eta=\eta_1} = F(\eta_1) + (\eta_1-\delta)\dfrac{\partial F}{\partial \eta}(\eta_1) > 0$

Since $b_4 > 0$, equation (B8) implies that the negative root of the characteristic equation (η_1) is unique.

The sign of $(xJ_2 - GJ_1)$:

Using the definitions of x and G we have;

(B9) $\quad xJ_2 - GJ_1 = - [\dfrac{U_{cm}\sigma}{U_c} + \dfrac{(\sigma-U_c)}{\delta'}\delta'' U_m]J_2 - [\dfrac{U_{cc}\sigma}{U_c} + (\dfrac{\sigma-U_c}{\delta'})\delta'' U_c]J_1$

Rewriting (B9)

(B10) $\quad xJ_2 - GJ_1 = (\dfrac{\sigma-U_c}{\delta'})\delta''[-U_m J_2 - U_c J_1] - \dfrac{\sigma}{U_c}[U_{cm}J_2 + U_{cc}J_1]$

Since $(\sigma-U_c)$ is negative at steady state, the first term on the right hand side of (B10) is negative. Using the definitions of J_1 and J_2, rewrite the second term on the right hand side of (B10) as,

(B11) $\quad -\dfrac{\sigma}{U_c}(U_{cm}J_2 + U_{cc}J_1) = \dfrac{-\sigma}{U_c}[U_{cm}(U_{cc}\dfrac{U_m}{U_c} - U_{cm}) + U_{cc}(U_{mm} - U_{cm}\dfrac{U_m}{U_c})]$

$= \dfrac{-\sigma}{U_c}[U_{cc}U_{mm} - U_{cm}^2]$

But since the utility function is strictly concave in c and m,

$$U_{cc}U_{mm} - U_{cm}^2 > 0$$

and thus the term in (B11) is negative, and therefore,

$$xJ_2 - GJ_1 < 0.$$